Office hours today — 11:20

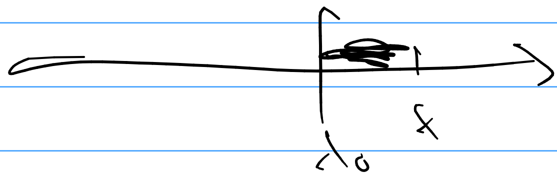
Recall: Taylor truncation error

$$\left| f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!}(x-x_0)^i \right| = O\left((x-x_0)^{n+1}\right)$$

$f(x_0)$

$f'(x_0)$

$f''(x_0)$



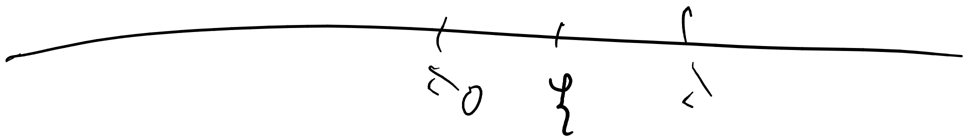
Taylor Remainders: the Full Truth

Let $f\colon \mathbb{R} \to \mathbb{R}$ be $n+1$-times differentiable on the interval (x_0, x) with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{“}C\text{”}} \cdot h^{n+1}.$$

$$O(h^{n+1})$$

$$\leq C \cdot h^{n+1}$$



In-class activity: Taylor series

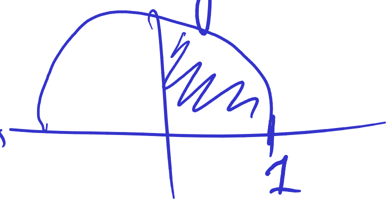
Using Polynomial Approximation

- Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3. = \tilde{f}(x)$$

How is that useful? Say, if I wanted the integral of f?

$f(x) = \sqrt{1-x^2}$

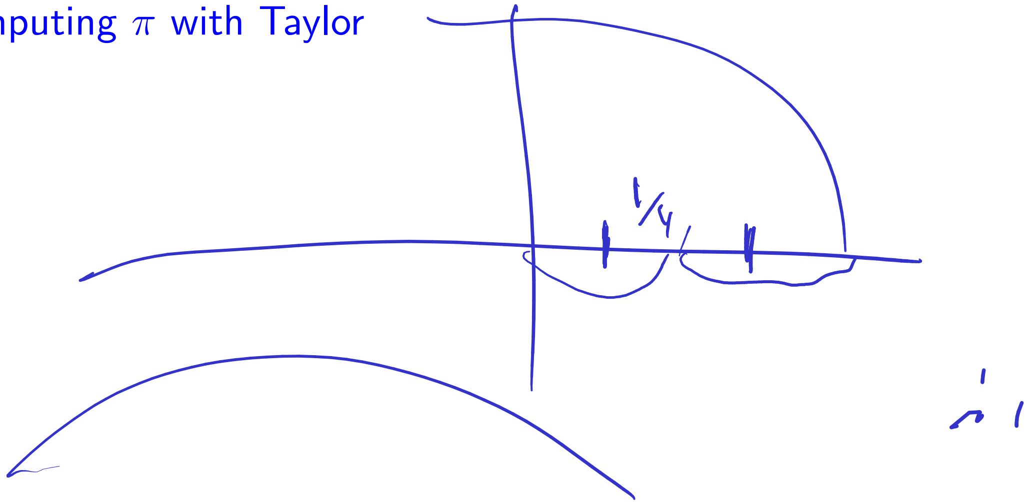


$$\int \tilde{f}(x)\,dx = \int a_0 + a_1 x + a_2 x^2 + a_3 x_3 \, dx$$

$$= \int a_0\,dx + a_1 \int x\,dx + \dots$$

$$\int_0^1 x^i\,dx = \frac{1}{i+1} \cdot \big(\underbrace{1^{i+1} - 0^{i+1}}_{1}\big)$$

Demo: Computing π with Taylor

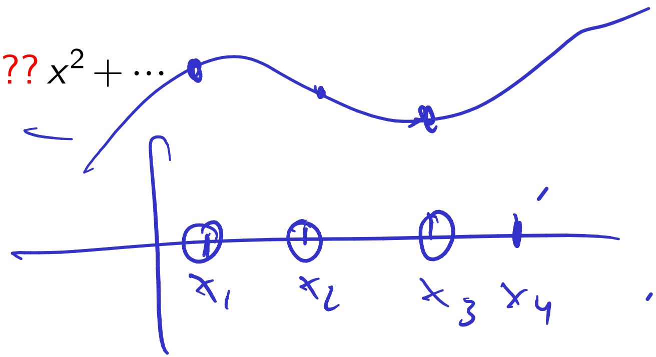


→ poly degree will be (# points) − 1

Reconstructing a Function From Point Values

- If we know function values at some points $f(x_1), f(x_2), \ldots, f(x_n)$, can we reconstruct the function as a polynomial?

$$f(x) = ??? + ???x + ???\,x^2 + \cdots$$



$$a_0 + a_1 \cdot x_1 + a_2 x_1^2 + \cdots + a_m x_1^m = f(x_1)$$

$$a_0 + a_1 \cdot x_2 + a_2 x_2^2 + \cdots + a_m x_2^m = f(x_2)$$

$$a_0 + a_1 \cdot x + a_2 x^2 + \cdots + a_m x^m$$

$$a_0 + a_1 \cdot x_n + a_2 x_n^2 + \cdots + a_m x_n^m = f(x_n)$$

$m+1$ unknowns

n equations.

want: $m+1 = n$

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & & & \vdots \\ 1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

Vandermonde matrix

coeffs

Demo: Polynomial Approximation with Point Values

Error in Interpolation

- What did we (empirically) observe about the error in interpolation in the demo?

  To fix notation: f is the function we're interpolating. $\tilde{f}$ is the interpolant that obeys $\tilde{f}(x_i) = f(x_i)$ for $x_i = x_1 < \ldots < x_n$. Let $h = x_n - x_1$ be the interval length.

- What is the error *at* the interpolation nodes?

  0 exactly

- Care to make an unfounded prediction? What will you call it?

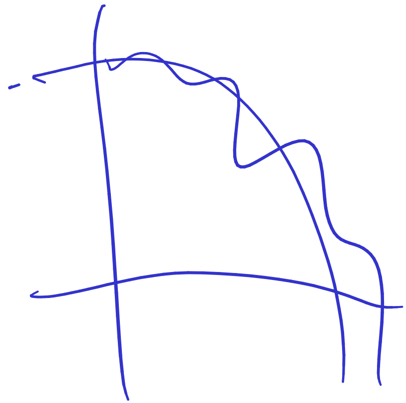
  $|f - (\text{approximating poly of degree } n)| = O(h^{n+1})$

Making Use of Interpolants

- Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1x + a_2x^2 + a_3x^3.$$

How is that useful? Say, if I wanted the integral of f?



Demo: Computing π with Interpolation

More General Functions

- Is this technique limited to the monomials $\{1, x, x^2, x^3, \ldots\}$?