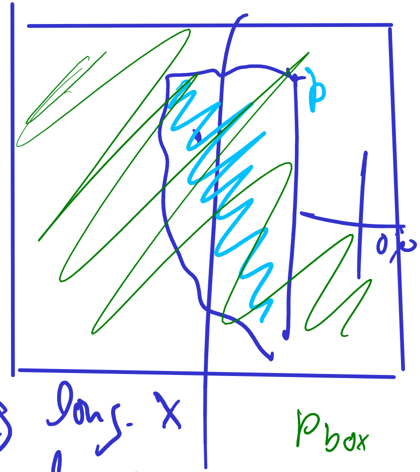
## Expected Value: Example II

- What is the expected snowfall in Illinois?

$$E[\underline{Snow}] = \int_{\mathbb{R}} snow(x)\, p(x)\, dx$$




x: longitude

snow(x): how much snow along long. x

p(x): how likely is a point on long. x to be in Illinois

$$\iint_{\mathbb{R}^2} 1 \cdot \begin{cases} 1 & \text{if } (x,y) \text{ in IL} \\ 0 & \text{if } (x,y) \text{ is not} \end{cases} = \text{Area}$$

$$E[\text{Snow}] = \int_{\mathbb{R}} \int_{\mathbb{R}} \text{Snow}(x,y)\, p(x,y)\, dx\, dy$$

$\text{Snow}(x,y)$ : how much snow at long $x$ and lat. $y$

$p(x,y)$ : how likely is it that long $x$ and lat $y$ is in IL

$$\rightarrow E[1] = \int_{\mathbb{R}} \int_{\mathbb{R}} 1\, p(x,y)\, dx\, dy = 1$$

$$\tilde{p}(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ is in IL} \\ 0 & \text{if not} \end{cases}$$

$$p(x,y) = C \cdot \tilde{p}(x,y)$$

If we can compute approximate expected values:

- use that to compute $E_{\tilde{p}}[1]$ using $\tilde{p}$. Use $E[1]=1$ to find $C$

$$C = \frac{1}{E_{\tilde{p}}(1)}$$

- Use that $E[\text{Snow}] = \iint_{R} \text{Snow}(x,y)\, p(x,y)\, d$

## Tool: Law of Large Numbers

Terminology:

- *Sample*: A random number $x_i$ whose values follow a distribution $p(x)$.

In words:

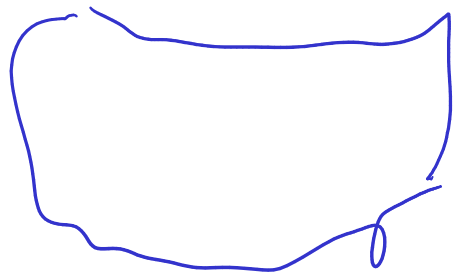
- As the number of samples $N \to \infty$, the average of samples converges to the expected value with probability 1.

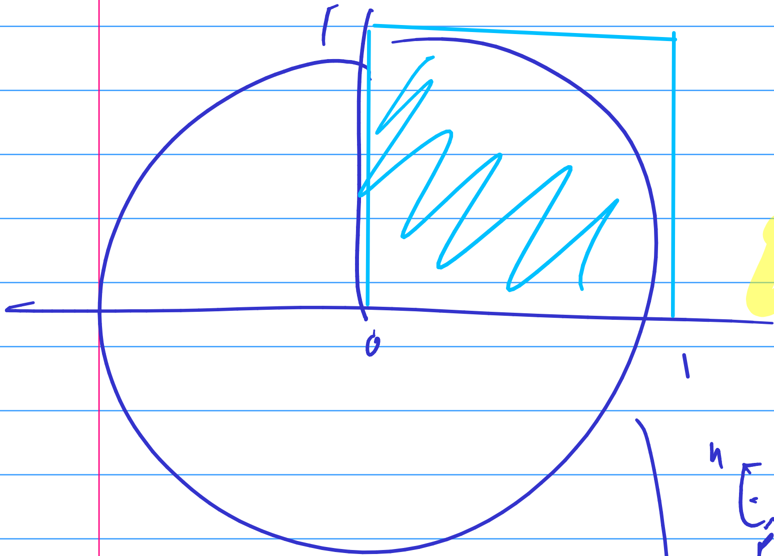
In symbols:

$$P\left[\lim_{N\to\infty} \frac{1}{N}\left(\sum_{n=1}^{N} x_i\right) = E[X]\right] = 1.$$

Or:

$$E[X] \approx \frac{1}{N}\left(\sum_{n=1}^{N} x_i\right)$$






$\pi$ = Area of unit circle

$$\hat{p}(x,y) = \begin{cases} 1 & \text{if } (x,y) \text{ in } \mathcal{D} \\ 0 & \text{if not} \end{cases}$$

(not yet PD

$$\mathbb{E}_{\hat{p}}[1] = \text{Area(box)} \cdot \mathbb{E}_{box}[1 \cdot \tilde{p}] = \underline{\text{Area of } \mathcal{D}}$$

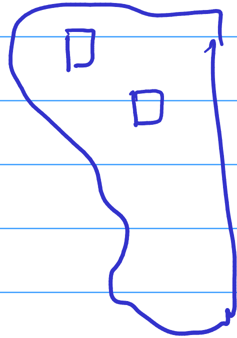
$$\text{Area} \cdot p(x,y) = \tilde{p}(x,y)$$

$$p(x,y) = \underline{\frac{1}{\text{Area of } \mathcal{D}} \cdot \tilde{p}(x,y)}$$

$$E_p(\text{Snow}) = \iint_{\mathbb{R}\,\mathbb{R}} \underline{\text{Snow}(x,y)}\ \underline{p(x,y)}\,dx\,dy \leftarrow$$

$$= \underline{\text{Area}(\text{box})} \int_{\mathbb{R}} \int_{\mathbb{R}} \underline{\text{Snow}(x,y)\,p(x,y)}\ \underline{p_{\text{uniform, box}}(x,y)}\ dx\,dy$$

$$= \text{Area}(\text{box}) \cdot E_{p_{\text{unif, box}}}[\text{Snow} \cdot p]$$



## Sampling: Approximating Expected Values

Integrals and sums in expected values are often challenging to evaluate.

- How can we approximate an expected value?

  **Idea:** Draw random samples. Make sure they are distributed according to $p(x)$.
- What is a Monte Carlo method?

## Sampling II: Approximating Expected Values

- What if I *can't* sample from $p(x)$?

  **Idea:** Draw uniformly distributed random samples.

**Demo:** Computing $\pi$ using Sampling
**Demo:** Errors in Sampling

## Sampling: Error

The Central Limit Theorem states that with

$$S_n := x_1 + x_2 + \cdots + x_n$$

for the $(x_i)$ independent and identically distributed we have that

$$\frac{S_n - n\,E[x_i]}{\sqrt{\sigma^2[x_i]n}} \to \mathcal{N}(0,1),$$

i.e. that term approaches the normal distribution. Or, short and imprecise:

$$\left|\frac{1}{n}S_n - E[x_i]\right| = O\left(\frac{1}{\sqrt{n}}\right).$$