Overview

1. - prop. of mat. norms
   - condition number of a matrix
   - la. solve

Recap: mat. norm

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$\|A\|_2 = \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2}$$

$$= \max_{\|x\|_2 = 1} \|A\vec{x}\|_2$$

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

prop. of a vector norm

- But also some more properties that stem from our definition:

$$\|A\vec{x}\| \leq \|A\| \|\vec{x}\|$$

$$\|AB\| \leq \|A\| \|B\|$$

submultiplicativity

$$\underbrace{\frac{\|Ax\|}{\|x\|}}_{\max} \leq \|A\|$$

Example: Orthogonal Matrices

- What is the 2-norm of an orthogonal matrix?

$$\|A\|_2 = \max_{\|\vec{x}\|_2=1} \|A\vec{x}\|_2 = \max_{\|\vec{x}\|_2=1} \|\vec{x}\|_2 = 1$$

$$\|Ax\|_2 = \sqrt{(Ax)_1^2 + \cdots + (Ax)_n^2}$$

$$= \sqrt{(Ax)\cdot(Ax)}$$

$$= \sqrt{(Ax)^T(Ax)}$$

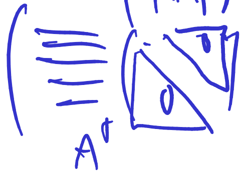
$$= \sqrt{x^T \underbrace{\cancel{A^T A}} x}$$

$$= \sqrt{x^T x} = \|x\|_2$$

$(\|A\|)$

$A^T A$ = diagonal

$(\|A\|)$



A^T

$A^T A = I \qquad AA^T = I$

orthogonal

$AB = I \qquad BA = I$

$A^T = A^{-1}$

$\|A x\| \le \|A\| \, \|x\|$

Conditioning

- Now, let's study condition number of solving a linear system

$$\underbrace{\overset{\text{~~input~~ input}}{A}\,x}_{\text{output}} = \overset{\text{input}}{b}.$$

cond # of solving

$$= \frac{\text{rel. error in output}}{\text{rel. error in input}} = \frac{\dfrac{\|\Delta x\|}{\|\vec{x}\|}}{\dfrac{\|\Delta b\|}{\|\vec{b}\|}}$$

$$= \frac{\|\Delta \vec{x}\| \; \|\vec{b}\|}{\|\vec{x}\| \; \|\Delta \vec{b}\|}$$

$$= \frac{\|A^{-1}\Delta \vec{b}\| \; \|A\vec{x}\|}{\|\vec{x}\| \cdot \|\Delta \vec{b}\|}$$

$$\le \frac{\|A^{-1}\| \, \cancel{\|\Delta b\|} \, \|A\| \, \cancel{\|x\|}}{\cancel{\|x\|} \cdot \cancel{\|\Delta b\|}}$$

$$= \|A^{-1}\| \, \|A\|$$

true $\vec{b}$

w/error: $\tilde{\vec{b}} = \vec{b} + \Delta \vec{b}$

true $\vec{x}$

w/error: $\tilde{\vec{x}} = \vec{x} + \Delta \vec{x}$

$\Delta \vec{x} = A^{-1} \Delta \vec{b}$

$\vec{b} = A\vec{x}$ ←

$A\tilde{x} = A(x + \Delta x)$

$= b + \Delta b$

- $\hookrightarrow$ <u>upper bound</u> on the condition number

… with more work: actually "sharp"

$\hookrightarrow$ best upper bound you can get

- $$\text{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

$$\text{cond}_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$$

condition number of a matrix is always relative to a given matrix norm.

- If A^{-1} doesn't exist, then

$A\vec{x}=b$ by convention $\text{cond}(A)=\infty$

Demo: Condition number visualized
Demo: Conditioning of 2×2 Matrices

Matrices with Great Conditioning (Part 1)

- Give an example of a matrix that is *very* well-conditioned.
  (I.e. has a condition-number that's *good* for computation.)
  What is the best possible condition number of a matrix?

Matrices with Great Conditioning (Part 2)

- What is the 2-norm condition number of an orthogonal matrix A?

In-class activity: Matrix Conditioning