Overview

- num. diff / finite differences
- num. int = quadrature
- eigen values

Demo: Taking derivatives with Vandermonde matrices

$$p(x) = \alpha_5 x^5 + \dots + \alpha_0 \cdot 1$$

$$p'(x) = \alpha_5 (5 \cdot x^4) + \dots + \alpha_0 \cdot 0$$

$$\to \quad p''(x) = \alpha_5 \cdot \varphi_5''(x) + \dots + \alpha_0 \cdot \varphi_0''(x)$$

$$\hookrightarrow \quad q(x) = \beta_5 \cdot \varphi_5(x) + \dots + \beta_0 \cdot \varphi_0(x)$$

$$\vec{f}' = V' \underbrace{V^{-1} \vec{f}}_{\vec{\alpha}}$$

$$V \vec{\alpha} = \vec{f}$$

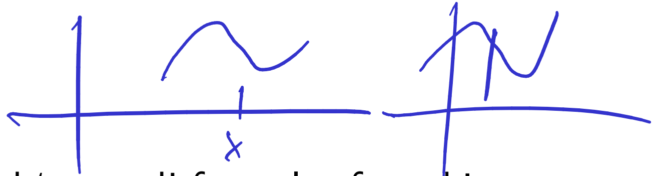
$$D = V' V^{-1}$$

$$D^2 = (V' V^{-1})(V' V^{-1})$$

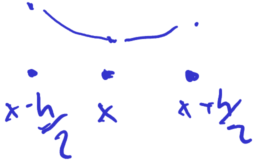
2 derivatives: 1 $\vec{f}'' = \underbrace{V'' V^{-1}}_{D_2} f$ 2

$$\vec{\beta} = V^{-1} D^2 \vec{f}$$

Finite Difference Formulas



- It is possible to use the process above to find 'canned' formulas for taking derivatives. Suppose we use three points equispaced points ($x - h$, x, $x + h$) for interpolation (i.e. a degree-2 polynomial).
  - What is the resulting differentiation matrix?
  - What does it tell us?



3×3

$$D = V' \, V^{-1}$$

w/ $(x - \frac{h}{2}, x, x + \frac{h}{2})$

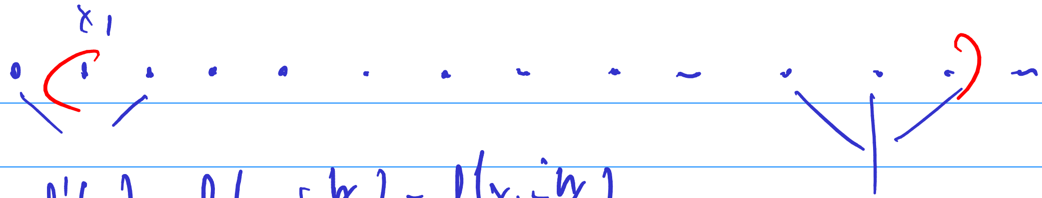
V, V' change if x changes

Expectation: D shouldn't change...

$$D \begin{pmatrix} f(x-\frac{h}{2}) \\ f(x) \\ f(x+\frac{h}{2}) \end{pmatrix} \approx \begin{pmatrix} f'(x-\frac{h}{2}) \\ f'(\cdot) \\ f'(x+\frac{h}{2}) \end{pmatrix}$$

Finite diff. formula →

$$\begin{bmatrix} & & \\ -1/h & & 1/h \\ & & \end{bmatrix} \begin{pmatrix} f(x-\frac{h}{2}) \\ f(x) \\ f(x+\frac{h}{2}) \end{pmatrix} = \begin{pmatrix} \sim \\ \frac{f(x+h/2) - f(x-h/2)}{h} \\ \sim \end{pmatrix} \approx f'(x)$$



$$f'(x_1) \approx \frac{f\left(x_1 + \frac{h}{2}\right) - f\left(x_1 - \frac{h}{2}\right)}{h}$$

Computing Integrals with Interpolation

- Can we use a similar process to compute (approximate) integrals of a function f?

$$f(x) \approx p(x) = \alpha_5 \cdot \varphi_5(x) + \dots + \alpha_0 \varphi_0(x)$$

$$\int_3^5 f(x)\,dx \approx \int_3^5 p(x)\,dx = \int \alpha_5 \cdot \varphi_5(x) + \dots + \alpha_0 \varphi_0(x)\,dx$$

$$= \alpha_5 \cdot \underbrace{\int_3^5 \varphi_5(x)\,dx}_{\int_3^5 x^5\,dx} + \dots + \alpha_0 \int_3^5 \varphi_0(x)\,dx$$

e.g. monomials

To compute $\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \alpha_i \underbrace{\int_a^b \varphi_i(x)\,dx}$

for n points

$$V\vec{\alpha} = \vec{f}$$
$$\vec{\alpha} = V^{-1}\vec{f}$$

$$\int_a^b f(x)\,dx \approx \underbrace{\begin{pmatrix} \int_a^b \varphi_0(x)\,dx \\ \vdots \\ \int_a^b \varphi_{n-1}(x)\,dx \end{pmatrix}}_{\vec{n}} \cdot \vec{\alpha} = \vec{n} \cdot \underbrace{V^{-1}\vec{f}}_{\vec{\alpha}} = \underbrace{\vec{n}^T V^{-1}}_{\vec{w}} \vec{f}$$

$$= \vec{w} \cdot \vec{f}$$

Example: Building a Quadrature Rule

Demo: Computing the Weights in Simpson's Rule

- Suppose we know

$$f(x_0) = 2 \qquad f(x_1) = 0 \qquad f(x_2) = 3$$

$$x_0 = \cancel{1}\, 0 \qquad x_1 = \frac{1}{2} \qquad x_2 = 1$$

How can we find an approximate integral? $\int_0^1 f(x)\,dx$

$$\int_0^1 1\,dx = 1 \qquad \int_0^1 x\,dx = \frac{1}{2} \qquad \int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$$

$$\vec{n} = \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \end{pmatrix} \qquad \underbrace{\vec{n}^T V^{-1}}_{\vec{w}}$$

$$= \frac{1}{6} \cdot \left(f(0) + 4 \cdot f\left(\frac{1}{2}\right) + f(1) \right)$$

$$\int_0^1 f(x)\,dx \approx \frac{1}{6} \cdot f(0) + \frac{4}{6} \cdot f\left(\frac{1}{2}\right) + \frac{1}{6} \cdot f(1)$$

Facts about Quadrature

- What does Simpson's rule look like on $[0, 1/2]$?

- What does Simpson's rule look like on $[5, 6]$?

- How accurate is Simpson's rule?
  Demo: Accuracy of Simpson's rule

10 Repeating Linear Operations: Eigenvalues and Steady States

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \mathbf{0}$ is called an eigenvector of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

- In that case, λ is called an eigenvalue.