Overview

- Eigenvalues
- What ev are good for
- SVD / lsq.

shift: $A-\sigma I \rightsquigarrow \lambda-\sigma$

invert: $A^{-1} \rightsquigarrow \lambda^{-1}$

Power Iteration

- What are the eigenvalues of A^{1000}?

  Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Further assume $\|\boldsymbol{x}_i\| = 1$.

unknown

pick at random $\rightarrow$ $\vec{y} = \alpha_1 \vec{x}_1 + \cdots + \alpha_n \vec{x}_n$

$A\vec{x}_i = \lambda_i \vec{x}_i$

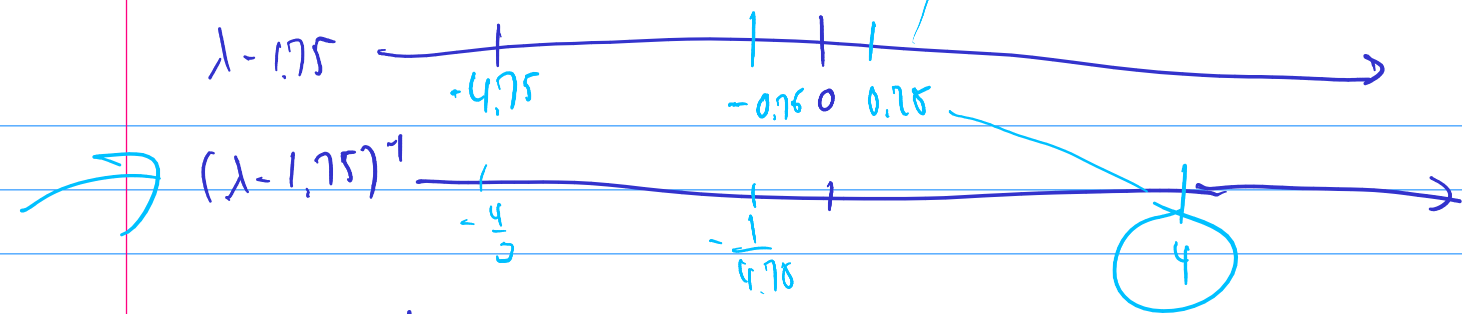
$$\frac{A^{1000}\vec{y}}{\lambda_1^{1000}} = \alpha_1 \frac{\lambda_1^{1000}}{\lambda_1^{1000}} \vec{x}_1 + \alpha_2 \frac{\lambda_2^{1000}}{\lambda_1^{1000}} \vec{x}_2 + \cdots + \alpha_n \frac{\lambda_n^{1000}}{\lambda_1^{1000}} \vec{x}_n$$

1   <1 <1

λ: -3, 0, 1, 2

$\frac{1}{\lambda}$: $-1/3$, $1/2$, 1

$\lambda+2$: -1, 0, 3, 4



$\underline{(A - \sigma I)^{-1}}$ ← power method that

The power method applied to $(A - \sigma I)^{-1}$ converges to the eigenvector (of A) that's closest to σ.

Inverse iteration

Can also choose the shift to be the Rayleigh quotient ← Rayleigh quotient it.

Power Iteration: Issues?

- What could go wrong with Power Iteration?

What about Eigenvalues?

- Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

Convergence of Power Iteration

- What can you say about the convergence of the power method? Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\|\boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)}\right\|.$$

$$e_2 \approx \left(\frac{\lambda_2}{\lambda_1}\right) e_1$$

$$e_3 \approx \left(\frac{\lambda_2}{\lambda_1}\right)^2 \qquad e_k \approx \left(\frac{\lambda_2}{\lambda_1}\right)^k e_1$$

Possible:
$|\lambda_2| = |\lambda_1|$ $\rightarrow$ power method won't converge :(

Scenario:
$\lambda_1 = 3$
$\lambda_2 = -3$

addressed by a shift

complex eigenvalues: use a complex shift

Inverse Iteration / Rayleigh Quotient Iteration

- Describe inverse iteration.

- Describe Rayleigh Quotient Iteration.

Demo: Power Iteration and its Variants
In-class activity: Eigenvalue Iterations

Computing Multiple Eigenvalues

- All Power Iteration Methods compute one eigenvalue at a time. What if I want *all* eigenvalues?

Deflation: $A\vec{v} = \lambda \vec{v}$

$V = \{\alpha \vec{v} : \alpha \in \mathbb{R}\}$

$A : V \to V$

$A : V^{\perp} \to V + V^{\perp}$

$A\vec{v} = \dots$?

$$A = \underbrace{\begin{pmatrix} \vec{v} & \text{Basis for } V^{\perp} \end{pmatrix}}_{Q} \begin{pmatrix} \lambda & ? \\ 0 & \boxed{B_1} \end{pmatrix} Q^T$$

eigenvalues of $A = \{\lambda\} \cup$ eigenvalues of B

Simultaneous Iteration

- What happens if we carry out power iteration on multiple vectors simultaneously?

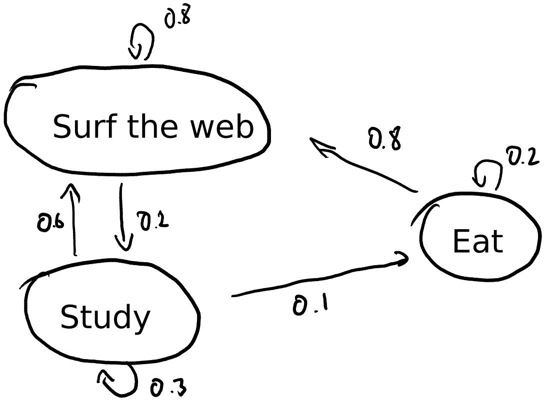
$A^k X \rightarrow$ all columns go toward the same vector !!

Idea: keep them different (at every iteration) using Gram-Schmidt

11 Eigenvalues: Applications

Markov chains

◦ Consider the following graph of states:



Suppose this is an accurate model of the behavior of the average student. :) How likely are we to find the average student in each of these states?