Overview

Example 7

- Eigenvalues : app
- SVD
- SVD : least squares

Understanding Time Behavior

- Many important systems in nature are modeled by describing the time rate of change of something.
  - E.g. every bird will have 0.2 baby birds on average per year.
  - But there are also foxes that eat birds. Every fox present decreases the bird population by 1 birds a year.
    Meanwhile, each fox has 0.3 fox babies a year. And for each bird present, the population of foxes grows by 0.9 foxes for every bird present.

Set this up as equations and see if eigenvalues can help us understand what's going on.

(t is measured in years)

$$\left.\begin{aligned} \frac{db}{dt} &= 0.2 \cdot b - f \\ \frac{df}{dt} &= 0.9\,b + 0.3\,f \end{aligned}\right] \sim \frac{d}{dt}\underbrace{\begin{pmatrix} b \\ f \end{pmatrix}}_{\vec{y}} = \begin{pmatrix} 0.2 & -1 \\ 0.9 & 0.3 \end{pmatrix}\begin{pmatrix} b \\ f \end{pmatrix}$$

$$\frac{d\vec{y}}{dt} = A\vec{y} \quad \xleftarrow{\text{plug in}} \quad \vec{y}(t) = e^{\lambda t}\vec{y}_0$$

assume

$$\lambda \cancel{e^{\lambda t}} y_0 = A \cancel{e^{\lambda t}} y_0 \longrightarrow A\vec{y} = \lambda \vec{y}$$

Demo: Understanding the birds and the foxes with eigenvalues
In-class activity: Eigenvalues 2

12 Approximate Undo: SVD and Least Squares

Singular Value Decomposition

$A = U \Sigma U^T$

- What is the Singular Value Decomposition ('SVD')?

$A = U \Sigma V^T$

$A : m \times n$

$U : \quad m \times m$ ✓ orthogonal? ← columns: left singular vec.

$\Sigma : \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \end{pmatrix}$ ✓

singular values

$m \times n$

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \cdots \geq \sigma_n \geq 0$ ✓

$V : \quad n \times n$ ✓ orthogonal ✓ ← columns: right singular vec.

Computing the SVD

- How can I compute an SVD of a matrix A?

eigenval. of A^TA

Find eigenval/vec. of $A^TAV = VD$

col.: eigenvectors of A^TA

$\to A^TA$ symmetric, pos. def.:

$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ real-valued, non-neg. col. of V (eigenvectors) are orth.

$\sigma_i = \sqrt{\lambda_i}$ $\qquad \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$

$A = U\Sigma V^T \rightsquigarrow \quad U = AV\Sigma^{-1}$

$U^TU = (AV\Sigma^{-1})^T AV\Sigma^{-1} = \Sigma^{-1}V^T\underbrace{A^TAV}\Sigma^{-1}$

$= \Sigma^{-1}\underbrace{\cancel{V^TV}}_{I}D\Sigma^{-1} = \Sigma^{-1}\Sigma^2\Sigma^{-1} = I$

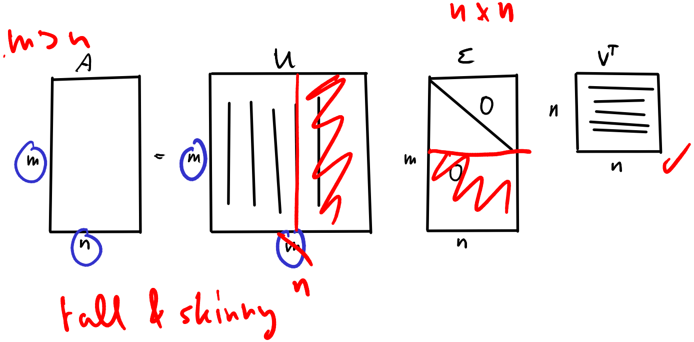
Demo: Computing the SVD

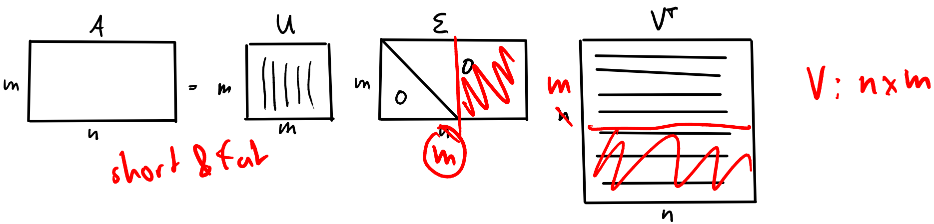
How Expensive is it to Compute the SVD?

Demo: Relative Cost of Matrix Factorizations

‘Reduced’ SVD

- Is there a ‘reduced’ factorization for non-square matrices?





13 SVD: Applications

13.1 Solving Funny-Shaped Linear Systems

Solve Square Linear Systems

- Can the SVD $A = U\Sigma V^T$ be used to solve *square* linear systems? At what cost (once the SVD is known)?

n^2

$$Ax = b$$

$$\Leftrightarrow \quad U\Sigma V^T \vec{x} = b \qquad \Big| \; U^T \cdot$$

$$\Sigma \underbrace{V^T \vec{x}}_{\vec{y}} = \underline{U^T b}$$

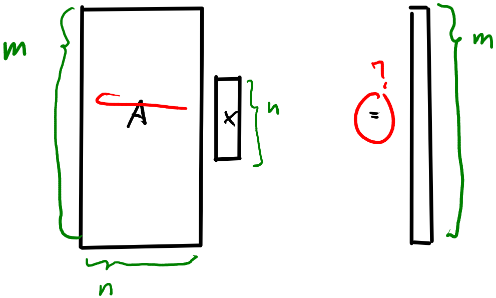
$$\begin{pmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{pmatrix} \vec{y} = \underbrace{U^T b} \qquad \leadsto \quad \vec{y} = V^T x$$

$$\to \quad V \vec{y} = \vec{x}$$

$\leadsto$ works, but: more expensive than fw/bw subst

Tall and Skinny Systems

- Consider a 'tall and skinny' linear system, i.e. one that has more equations than unknowns:



$Ax = b$ ← nope.

$\|Ax - b\|_2$ ← make as small as possible

residual
$\vec{r} = A\vec{x} - \vec{b}$

In the figure: $m > n$. How could we solve that?

minimize $\|Ax - b\|_2$

equiv. minimize $\|Ax - b\|_2^2 = \|r\|_2^2 = r_1^2 + \dots + r_m^2$

This is called a least squares problem: $A\vec{x} \cong \vec{b}$

Find $\vec{x}$ so that $\|A\vec{x} - \vec{b}\|_2^2$ is as small as possible.

Solving Least-Squares

- How can I actually *solve* a least-squares problem $A\boldsymbol{x} \cong \boldsymbol{b}$?

minimize $\|Ax-b\|_2^2$

$$= \|U\Sigma V^T\vec{x}-\vec{b}\|_2^2$$

$$= \|U^T(U\Sigma V^T\vec{x}-\vec{b})\|_2^2$$

$$= \|\Sigma \underbrace{V^T\vec{x}}_{\vec{y}} - U^T\vec{b}\|_2^2$$

$\vec{y} \rightsquigarrow V^Tx=y$

$A = U\Sigma V^T$

For any $\vec{y}$:

$\|U\vec{y}\|_2 = \|y\|_2$

$\|Uy\|_2 \le \|y\|_2$

minimize

$$\begin{pmatrix} \sigma_1 & & & 0 \\ & \ddots & & \\ & & \sigma_k & \\ & & & 0 \\ & 0 & & \ddots \\ & & & & 0 \end{pmatrix} \vec{y} \cong U^T\vec{b}$$

residual 0 if

$\neq 0$

0

$$y_i = \begin{cases} (U^Tb)_i/\sigma_i & 1 \le i \le k \\ 0 & k+1 \le i \le n \end{cases}$$

$\sigma_i \neq 0$

$\vec{y}$ figured out

$\to$ find $\vec{x} = V\vec{y}$.

In-class activity: SVD and Least Squares

Solve $A\vec{x} \cong \vec{b}$

1. Get $A = U\Sigma V^T$

2. Solve $\Sigma\vec{y} = U^T\vec{b}$

3. $\vec{x} = V\vec{y}$

"fake inverse" $\Sigma^+ = \begin{pmatrix} 1/\sigma_1 & & & 0 \\ & \ddots & & \\ & & 1/\sigma_k & \\ 0 & & & 0 \ddots 0 \end{pmatrix} \rightsquigarrow \vec{x} = \underbrace{U^T\Sigma^+V}_{\text{pseudo inverse}}\vec{y}$

The Pseudoinverse: A Shortcut for Least Squares

- How could the solution process for $A\boldsymbol{x} \cong \boldsymbol{b}$ be with an SVD$A = U\Sigma V^T$ be 'packaged up'?

The Normal Equations

- You may have learned the 'normal equations' $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$ to solve $A\boldsymbol{x} \cong \boldsymbol{b}$. Why not use those?