Overview

- Least squares / pseudo inv.
- Data fitting
- Norms and condition #s
- Low-rank approximation

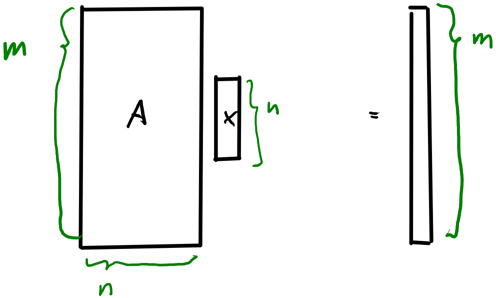
Tall and Skinny Systems

$A = UDU^{-1}$ ← doesn't always exist

$A = U\Sigma V^T$ ← always exists

orth diag orth

- Consider a 'tall and skinny' linear system, i.e. one that has more equations than unknowns:



In the figure: $m > n$. How could we solve that?

$Ax \cong b$

$\underline{\|Ax-b\|_2^2}$

$$\|Ax-b\|_2^2 = \|U\Sigma V^T x - b\|_2^2 = \|\cancel{U^T}(\cancel{U}\Sigma V^T x - b)\|_2^2$$

want: small

$$= \overbrace{\|\Sigma \underbrace{V^T x}_{y} - U^T b\|_2^2}$$

$$\begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \\ & & 0 & & & \end{pmatrix} y \cong \begin{pmatrix} z_1 \\ \vdots \\ z_{k+1} \\ \vdots \\ z_m \end{pmatrix}$$

$y_1 = z_1/\sigma_1$

$\vdots$

$y_k = z_k/\sigma_k$

$y_{k+1} = 0 \;\cdots\; y_n = 0$

pseudo inverse of Σ

$$\Sigma^+ = \begin{array}{c} \\ n \end{array} \overset{m}{\begin{pmatrix} 1/\sigma_1 & & & & & \\ & \ddots & & & & \\ & & 1/\sigma_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}} \begin{array}{l} \\ \\ \\ \leftarrow k+1 \\ \\ \leftarrow n \end{array}$$

Intention: $\vec{y} = \overset{n \times m}{\Sigma^+} \vec{z}$

$= \Sigma^+ (U^T b)$

$V^T \vec{x} = \vec{y} \rightsquigarrow \vec{x} = V \vec{y}$

$\underset{n}{\Sigma} \overset{n}{\vec{y}} = \overset{m}{\vec{z}}$

$$\vec{x} = \underbrace{V \Sigma^+ U^T}_{A^+} \vec{b}$$

$A^+ \leftarrow$ pseudo inverse of A

Solving Least-Squares

- How can I actually *solve* a least-squares problem $A\boldsymbol{x} \cong \boldsymbol{b}$?

In-class activity: SVD and Least Squares

The Pseudoinverse: A Shortcut for Least Squares

- How could the solution process for $A\boldsymbol{x} \cong \boldsymbol{b}$ be with an SVD$A = U\Sigma V^T$ be 'packaged up'?

The Normal Equations

- You may have learned the 'normal equations' $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ to solve $A\boldsymbol{x} \cong \boldsymbol{b}$. Why not use those?

$A\vec{x} \cong \vec{b}$

$A^TA\vec{x} = A^T\vec{b}$

"normal" equations

$\text{cond}(A^TA) \le \text{cond}(A) \cdot \underbrace{\text{cond}(A^T)}_{=\text{cond}(A)} \approx \text{cond}(A)^2$

$\text{cond}(AB) \le \text{cond}(A)\,\text{cond}(B)$

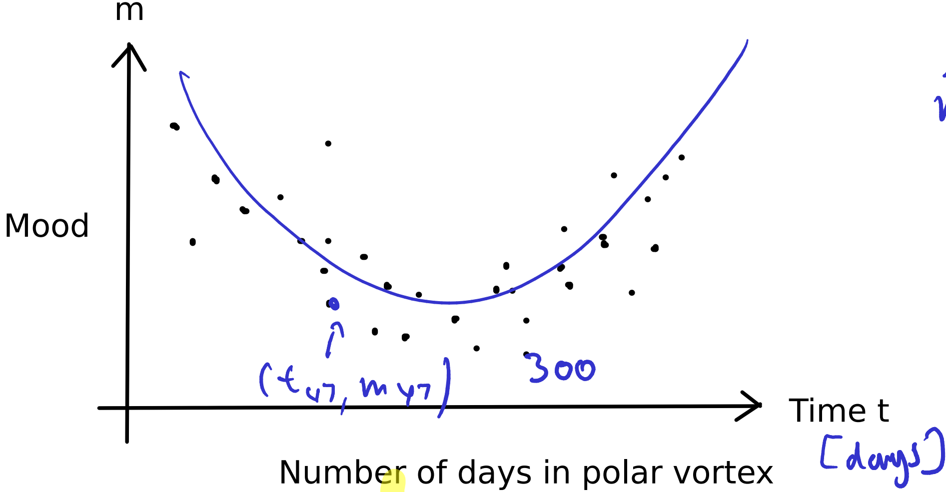
↳ tend to be poorly conditioned

↳ let's not

13.2 Data Fitting

Fitting a Model to Data

- How can I fit a model to measurements? E.g.:



$$\hat{m}(t) = \alpha + \beta t + \gamma t^2$$

$$\hat{m}(t_1) = \alpha + \beta t_1 + \gamma t_1^2 \approx m_1$$

$$\vdots$$

$$\hat{m}(t_{47}) = \alpha + \beta t_{47} + \gamma t_{47}^2 \approx m_{47}$$

$$\vdots$$

$$\hat{m}(t_{300}) = \alpha + \beta t_{300} + \gamma t_{300}^2 \approx m_{300}$$

$$\left(\begin{pmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \\ 1 & t_{47} & t_{47}^2 \\ \vdots & \vdots & \\ 1 & t_{300} & t_{300}^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \approx \vec{m} \right.$$

Demo: Data Fitting using Least Squares

13.3 Norms and Condition Numbers

$$\|Ux\|_2 = \|x\|_2$$

Meaning of the Singular Values

- What do the singular values mean? (in particular the first/largest one)

biggest

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^T$$

$$\|A\|_2 = \|U \Sigma V^T\|_2 = \max_{\|x\|_2 = 1} \|U \Sigma V^T x\|_2 = \max \|\Sigma \underbrace{V^T x}_{y}\|_2$$

$$= \max_{\|V^T x\| = 1} \|\Sigma V^T x\|_2 = \max_{\|y\|_2 = 1} \|\Sigma y\|_2$$

$$= \|\Sigma\|_2 = \sigma_1$$

Condition Numbers

- How would you compute a 2-norm condition number?

$$A = U \Sigma V^T = \underline{U} \overbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}}^{\sigma_i \geq 0} \underline{V^T}$$

$$\text{cond}(A) = \|A\| \, \|A^{-1}\| = \sigma_1 / \sigma_n$$

$$A^{-1} = V \Sigma^{-1} U^T \Leftarrow V \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{pmatrix} U^T$$

If $\sigma_n = 0 \Rightarrow A$ not invertible $\Rightarrow \text{cond}(A) = \infty$

13.4 Low-Rank Approximation

SVD as Sum of Outer Products

- What's another way of writing the SVD?