Overview

$x^2 + \alpha x + y$

Low-rank approximation

Iteration

Equation solving

SVD as Sum of Outer Products

- What's another way of writing the SVD?

$\vec{u} = \begin{pmatrix} \vdots \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} \vdots \end{pmatrix}$

$\operatorname{rank}(u v^T) = 1$

$$A = U \Sigma V^T = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} - & \sigma_1 v_1 & - \\ & \vdots & \\ - & \sigma_n v_n & - \end{pmatrix}$$

inner product

$\vec{u}^T \vec{v} = \underbrace{\text{one}}$ number

outer product

$\vec{u}\, v^T =$

$\vec{u} \qquad v^T$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 10 & 20 & 30 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 20 & 40 & 60 \\ 30 & 60 & 90 \end{pmatrix}$$

$$= \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \cdots + \sigma_n u_n v_n^T$$

Low-Rank Approximation (I)

- What is the *rank* of $\underline{\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T}$?   1

- What is the *rank* of $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T$?   2

Demo: Image Compression

Low-Rank Approximation

- What can we say about the low-rank approximation

$$A_k = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^T$$

to

$$A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + \sigma_1 \boldsymbol{u}_n \boldsymbol{v}_n^T ?$$

k+1 SVD

For simplicity, assume $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$.

Then among all rank-k matrices B, A_k is the one that satisfies

$$\min_B \|A - B\|_2 = \|A - A_k\|_2$$

Also: $\|A - A_k\|_2 = \|\sigma_{k+1} u_{k+1} v_{k+1}^T + \cdots \sigma_n u_n v_n^T\|_2 = \sigma_{k+1}$

~~[Spring 17: Add PCA]~~

$$\|A - A_k\|_F = \sqrt{\sigma_{k+1}^2 + \cdots + \sigma_n^2}$$

Part 3:
Approximation–When the Exact Answer is Out of Reach

14 Iteration and Convergence

- What is linear convergence? quadratic convergence?

for power it:

$$\|e_{k+1}\| = \left|\frac{\lambda_2}{\lambda_1}\right| \|e_k\|$$

$$\|e_{k+1000}\| = \left(\frac{\lambda_2}{\lambda_1}\right)^{1000} \|e_k\| \leftarrow \text{linear convergence}$$

gains a fixed - possibly fractional - number of digits every time

$$\|e_{k+1}\| \lesssim C \cdot \|e_k\|$$

$$\|e_{k+1}\| \le \underline{C} \cdot \|e_k\|^2 \leftarrow \text{quadratic convergence}$$

Example for quadratic:

$C = 1$

$\|e_1\| = 0.1 \sim 1$

$\|e_2\| = 0.01 = 10^{-2}$ ~ 2 digits

$\|e_3\| = 10^{-4}$ → 4 digits

$\|e_4\| = (10^{-4})^2 = 10^{-(2\cdot 4)} = 10^{-8}$

An iterative method converges with rate r if

$$\lim_{k \to \infty} \frac{\|e_{k+1}\|}{\|e_k\|^r} = C \begin{cases} > 0 \\ < \infty \end{cases}$$

convergent with rate 1 : linear

2 : quadratic

About Convergence Rates

Demo: Rates of Convergence

- Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

15 Solving One Equation

Solving Nonlinear Equations

- What is the goal here?

$ax + b = 0$

$a\underline{x^2} + bx + c = 0$

Given: equation

$f(x) = 0$ Find x so that the equation is true.

$f(x) = y$ $\tilde{f}(x) = f(x) - y$

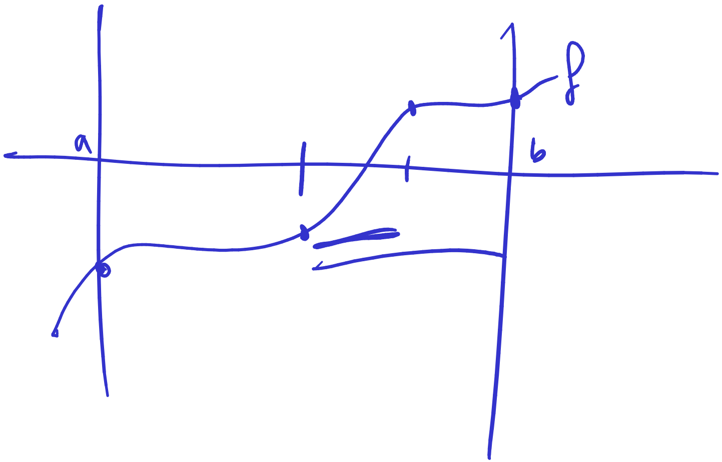
$\downarrow$ solve

$\tilde{f}(x) = 0$

Bisection Method

Demo: Bisection Method

- What's the rate of convergence? What's the constant?



If:

$f(a)$ has a different sign than $f(b)$

$f(a) > 0$

$f(b) < 0$

Newton's Method

- Derive Newton's method.