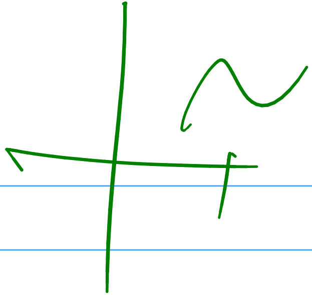
Overview

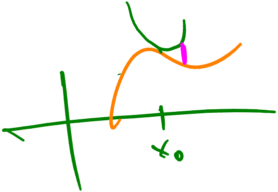
- Taylor error
- App. of poly
- Interpolation
  (give me a poly from f values)



Errors in Taylor Approximation (I)

Can't sum infinitely many terms. Have to ➧ truncate. How big of an error does this cause?

Demo: Polynomial Approximation with Derivatives (Part II)



$$\left| f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right|$$

$$\approx C \cdot h^{n+1}$$

Taylor Remainders: the Full Truth

Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-times differentiable on the interval (x_0, x) with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

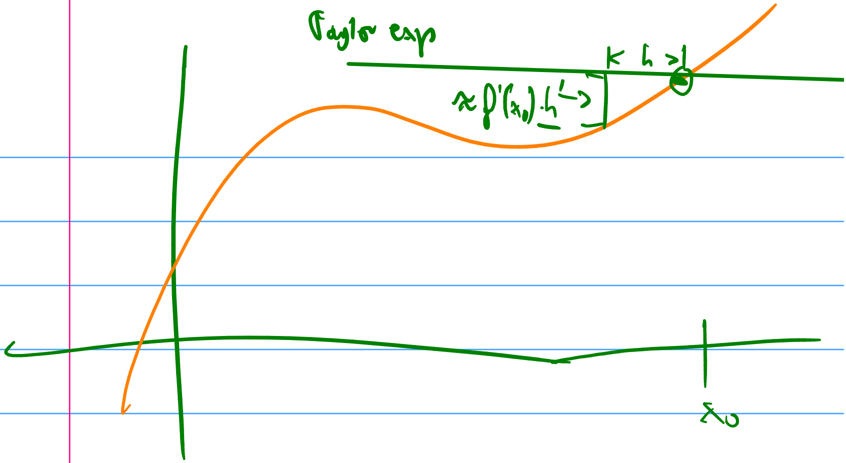
Xi

$$f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{“}C\text{”}} \cdot (\xi - x_0)^{n+1}$$

and since $|\xi - x_0| \leqslant h$

$$\left| f(x_0+h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leqslant \underbrace{\frac{\left|f^{(n+1)}(\xi)\right|}{(n+1)!}}_{\text{“}C\text{”}} \cdot h^{n+1}.$$

0th order
Taylor exp



Intuition for Taylor Remainder Theorem

Given the value of a function and its derivative $f(x_0), f'(x_0)$, prove the Taylor error bound.

$$\to \quad f(x) = f(x_0) + \int_{x_0}^{x} f'(w_0)\, dw_0$$

$$f'(w_0) = f'(x_0) + \int_{x_0}^{w_0} f''(w_1)\, dw_1$$

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(w_0)\, dw_0$$

$$= f(x_0) + \int_{x_0}^{x} f'(x_0) + \int_{x_0}^{w_0} f''(w_1)\, dw_1\, dw_0$$

$$= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x}\int_{x_0}^{w_0} f''(w_1)\, dw_1\, dw_0$$

In-class activity: Taylor series

$$\left| f(x) - \left(f(x_0) + f'(x_0)(x - x_0)\right) \right|$$

$$\leq \underline{\max_{\xi \in [x, x_0]} |f''(\xi)|} \cdot \underline{\frac{(x - x_0)^2}{2}}$$

$$\text{Error}(h) \approx C \cdot h^{n+1}$$

$$C \cdot h^4$$

Using Polynomial Approximation

Suppose we can approximate a function as a polynomial:

$$f(x) \approx \overbrace{a_0 + a_1 x + a_2 x^2 + a_3 x^3}^{p(x)}.$$

How is that useful?

E.g.: What if we want the integral of f?

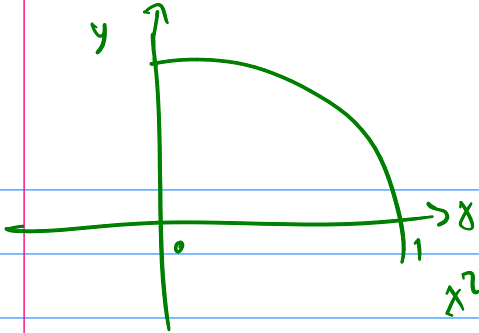
$\to \exp\left(1/x^2 + \sin x\right)$

$$\int_0^1 f(x)\,dx \approx \int_0^1 p(x)\,dx$$

$$= \int_0^1 a_0 + a_1 x + a_2 x^2 + a_3 x^3\,dx$$

$$= \left(a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right)_0^1$$

Demo: Computing π with Taylor



$y = \sqrt{1-x^2}$

$x^2 + y^2 =$

Reconstructing a Function From Point Values

If we know function values at some points $f(x_1), f(x_2), \ldots, f(x_n)$, can we reconstruct the function as a polynomial?

$$f(x) = \underbrace{\overset{a_0}{???} + \overset{a_1}{???}x + \overset{a_2}{???}x^2 + \cdots}_{p(x)}$$

called "Interpolation"

$x_1 \to p(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots = f(x_1)$

$x_2 \to p(x_2) = a_0 + a_1 x_2 + a_2 x_1^2 + \cdots = f(x_2)$

$\vdots$

$x_n \to p(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots = f(x_n)$

$\hookrightarrow Ax = b$

$$
\begin{array}{cc}
 & \overbrace{\phantom{1 \quad x_1 \quad x_1^2 \cdots x_1^{m-1}}}^{m} \\
n & \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{m-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{m-1} \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{m-1} \end{pmatrix}
\end{array}
\begin{pmatrix} a_0 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}
$$

Guess: pick $n = m$

Vandermonde matrices

Vandermonde Linear Systems

Polynomial interpolation is a critical component in many numerical models.

Demo: Polynomial Approximation with Point Values

Error in Interpolation

How did the interpolation error behave in the demo?

To fix notation: f is the function we're interpolating. $\tilde{f}$ is the interpolant that obeys $\tilde{f}(x_i) = f(x_i)$ for $x_i = x_1 < \ldots < x_n$. $h = x_n - x_1$ is the interval length.

What is the error *at* the interpolation nodes?

Care to make an unfounded prediction? What will you call it?

Proof Intuition for Interpolation Error Bound

Let us consider an interpolant $\tilde{f}$ based on $n = 2$ points so

$$\tilde{f}(x_1) = f(x_1) \quad \text{and} \quad \tilde{f}(x_2) = f(x_2).$$

Prove the interpolation error bound in this case.