Overview

$$\|\tilde{x} - x_0\|_5$$

Norms
- vector
- matrix
- conditioning

1D:
abs. error:

$$|\tilde{x} - x_0|$$

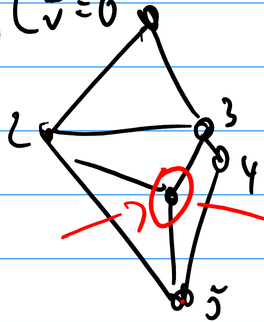
$$L\vec{v} = \vec{0}$$

$$\begin{array}{cc} & \\ i & \begin{pmatrix} -1 & -1 & & 4 & -1 & -1 \end{pmatrix} \end{array} v = \begin{pmatrix} 0 \end{pmatrix}$$

$$-\sum_{j:\, i \Leftrightarrow j \text{ connected}} v_j + d\, v_i = 0$$

$$V_i = \frac{1}{\#(\text{nodes conn to } i)} \sum_{j:\ \text{connected to } i} V_j$$

Assuming $L\vec{v} = \vec{0}$



value here: take the average of nb.

Norms

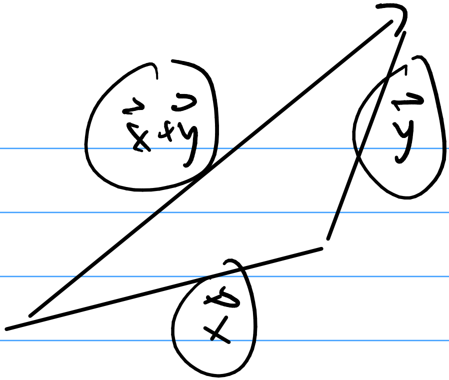
What's a norm?

$$\vec{x} \in \mathbb{R}^n \qquad \|\vec{x}\| \geq 0$$

Define norm.

- $\|\vec{x}\| = 0 \Rightarrow \vec{x} = \vec{0}$
- $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$
- $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

triangle inequality


$\vec{x}+\vec{y}$
$\vec{y}$
$\vec{x}$

Examples of Norms

> What are some examples of norms?

$$\|\vec{x}\|_2 = \left\|\begin{pmatrix}1\\2\\3\end{pmatrix}\right\| = \sqrt[2]{1^2+2^2+3^2}$$

$$\|\vec{x}\|_p = \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_n|^p}$$

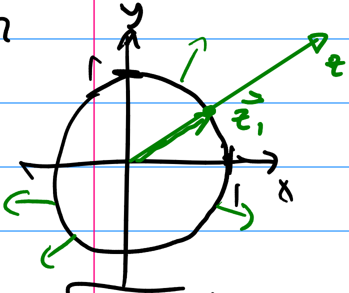
$p = 3$

$p \geq 1$ or $p = \infty$

unit ball

Draw: $\|\vec{x}\|_p = 1$

$p=2$



$\sqrt{x^2+y^2}=1$

$$\|z\|_2 = \|\alpha z_1\|_2 = |\alpha| \underbrace{\cancel{\|z_1\|_2}}_{1}$$

$$\rightarrow \vec{z} = \alpha \vec{z}_1$$

$$\|\vec{z}_1\|_2 = 1$$

Find norm from unit ball:

- Find z_1 ; $\vec{z} = \alpha \vec{z}_1$, $\|z_1\|=1$
- $\|z\| = |\alpha|$

$$\frac{\vec{x}}{\|\vec{x}\|} \rightsquigarrow \left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = \left\| \frac{1}{\|\vec{x}\|} \cdot \vec{x} \right\|$$

$$= \frac{1}{\|x\|} \cdot \|x\| = 1$$

normalizing a vector

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2 = \sqrt{0{,}1\, x^2 + y^2}$$

Demo: Vector norms

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

Absolute and Relative Error

What are the absolute and relative errors in approximating the location of Siebel center $(40.114, -88.224)$ as $(40, -88)$ using the 2-norm?

Demo: Calculate geographic distances using `tripstance.com`

Matrix Norms

What norms would we apply to matrices?

$$\|Ax\| \le \underline{\|A\|}\,\|x\|$$

new: matrix norm

want:

after

before

$$\frac{\|Ax\|}{\|x\|} \le \underline{\|A\|}$$

for every possible x

$$\underline{\|A\|} = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$$

same v. n.

matrix norm

defined in terms of vector norm

$$\|A\| = \max_{\substack{\vec{x} \in \mathbb{R}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|}{\|\vec{x}\|} = \max \frac{1}{\|\vec{x}\|} \|Ax\|$$

$$= \max \Big\| A \underbrace{\frac{\vec{x}}{\|\vec{x}\|}}_{\|\uparrow\| = 1} \Big\|$$

Demo: Matrix norms

In-class activity: Matrix norms

10 nonzeros / row

n rows

10n total nonzeros

→ sum : 10

→ ntimes: 10n

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition: