Overview

- Norm
  - ↳ vector
  - ↳ matrix
  - ↳ conditioning
  - ↳ guts of la. solve()

$\vec{x} : \mathbb{R}^n \to \mathbb{R}_0^+$

$\|\vec{x}\| \geq 0$

$\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = 0$

$\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|$

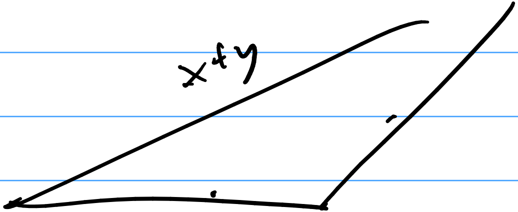
$\underline{\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|}$

triangle ineq.

$p \geq 1$

$\|\vec{x}\|_p = \sqrt[p]{\sum |x_i|^p}$

$p = \infty$


x+y

Matrix Norms

What norms would we apply to matrices?

$$\|A\|_F = \sqrt{\sum_{i,j} |A_{i,j}|^2}$$

"Frobenius"

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

corresponding $\|\cdot\|$ vec norm

"induced" from a vector norm

$$\begin{pmatrix} 2 & \\ & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ y \end{pmatrix}$$

$$\| \quad \|_2 \; ?$$

$$\max_{(x,y) \neq (0,0)} \frac{\sqrt{4x^2 + y^2}}{\sqrt{x^2 + y^2}}$$

$$= \max_{x^2 + y^2 = 1} \sqrt{4x^2 + y^2} = 2$$

$$n\times 1 \left\| \overset{A}{\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}} \right\|$$

$$\max_{\|x\|=1} \|A\overset{(x_1)}{x}\| = \max_{x_1 \in \{-1,1\}} \left\| \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} x_1 \right\|_1$$

$$= \left\| \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\|_1 = 9$$

$$\vec{x} = (x, y, z)$$

$$1 \times n \qquad A = \left\| \begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \right\|_1$$

$$\|A\|_1 = \max_{\|\vec{x}\|_1 = 1} |2x + 3y + 4z|$$

$$|x| + |y| + |z| = 1 \qquad = 4$$

$$0 \quad 0 \quad 1$$

Demo: Matrix norms

In-class activity: Matrix norms

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$.
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition:

$$\|A\vec{x}\| \leq \|A\| \, \|\vec{x}\|$$

$$\|Ax\| \leq \|A\| \, \|x\|$$

submultiplicativity

Example: Orthogonal Matrices

What is the 2-norm of an orthogonal matrix?

$$Q^T Q = I$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$$\|\vec{x}\|_2 = \sqrt{\sum x_i^2} = \sqrt{\vec{x}\cdot\vec{x}}$$

$$\to (\vec{x},\vec{y}) = \sum_{i=1}^{n} x_i y_i$$

$$\to \vec{x}\cdot\vec{y}$$

$$\|Q\|_2 = \max_{\|\vec{x}\|_2 = 1} \|Q\vec{x}\|_2$$

$$= \max_{\|\vec{x}\|_2 = 1} \sqrt{(Q\vec{x}) \cdot Q\vec{x}}$$

$$= \max_{\|x\|_2 = 1} \sqrt{x^T \underbrace{Q^T Q}_{I} \vec{x}}$$

$\vec{x} \cdot \vec{y} = x^T y$

$$= \max_{\|x\|_2 = 1} \sqrt{\vec{x}^T \vec{x}} = \max_{\|\vec{x}\|_2 = 1} \|\vec{x}\|_2$$

$$= 1$$

Conditioning

Now, let's study condition number of solving a linear system

$$Ax = b.$$

$\tilde{x}$

$\tilde{b}$

$$\text{Cond(solve)} = \frac{\text{rel err output}}{\text{rel. err. input}}$$

$x_0 + \Delta x$

$b_0 + \Delta b$

true val. Errors

$$A(x_0 + \Delta x) = b_0 + \Delta b$$

$$\Rightarrow A \Delta x = \Delta b$$

$$\Delta x = A^{-1} \Delta b$$

$$\text{cond} \approx \frac{\|\Delta x\|}{\|x_0\|} \Big/ \frac{\|\Delta b\|}{\|b_0\|}$$

$$= \frac{\|\Delta x\|}{\|x_0\|} \cdot \frac{\|b_0\|}{\|\Delta b\|} \leftarrow$$

$$= \frac{\|A^{-1}\Delta b\| \cdot \|A x_0\|}{\|x_0\| \, \|\Delta b\|}$$

$$\le \frac{\|A^{-1}\| \, \cancel{\|\Delta b\|} \cdot \|A\| \, \cancel{\|x_0\|}}{\cancel{\|x_0\|} \, \cancel{\|\Delta b\|}}$$

$$= \|A\| \, \|A^{-1}\| \geq 1$$

$$1 = \|I\| = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\|$$

with some extra work:

Can show

$$\text{cond} = \|A\| \, \|A^{-1}\|$$

Demo: Condition number visualized

Demo: Conditioning of 2×2 Matrices

$$\begin{pmatrix} 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = |2x + 3y + 4z|$$

$$\max \frac{\|Ax\|_1}{\|x\|_1}$$

$$\mathrm{cond}(A y = b) = \|A\| \|A^{-1}\|$$

$$\mathrm{cond}(A x = b) =$$

$$\mathrm{cond}(A^{-1} b = x) =$$

$$\|A^{-1^{-1}}\| \|A^{-1}\|$$

$$\|A\| \|A^{-1}\|$$

|rel. err output|

$\leq$ cond $\cdot$ |rel. err. in input|

$$\mathrm{cond}(A \cdot x = b) = \|A\| \cdot \|A^{-1}\|$$

$$Ax = b \iff A^{-1}b = x$$

$$\mathrm{cond}(A^{-1}b \quad x)$$
$$= \|A^{-1^{-1}}\| \, \|A^{-1}\|$$

More Properties of the Condition Number

What is $\operatorname{cond}(A^{-1})$?

What is the condition number of applying the matrix-vector multiplication $A\boldsymbol{x} = \boldsymbol{b}$? (I.e. now $\boldsymbol{x}$ is the input and $\boldsymbol{b}$ is the output)

Matrices with Great Conditioning (Part 1)

Give an example of a matrix that is *very* well-conditioned.
(I.e. has a condition-number that's *good* for computation.)
What is the best possible condition number of a matrix?