Overview

Eigenvalues/PI
  Diagonalization
  Error
 PI refinements
Eigenvalues apps.

$$|\lambda_1| > |\lambda_2| \cdots$$

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

$$\frac{A^{1000} x}{\lambda_1^{1000}} = \underbrace{\alpha_1 x_1}_{} + \underbrace{\alpha_2 \frac{\lambda_2^{1000}}{\lambda_1^{1000}}}_{} + \alpha_n \frac{\lambda_n^{1000}}{\lambda_1^{1000}} x$$

Diagonalizability

If we have n eigenvectors with different eigenvalues, the matrix is diagonalizable.

$$X = \begin{pmatrix} | & & | \\ \vec{x}_1 & \cdots & \vec{x}_n \\ | & & | \end{pmatrix}$$

$$A\vec{x}_i = \lambda_i \vec{x}_i$$

$$\vec{x}_i \neq 0$$

$$AX = XD \qquad \cancel{DX} \qquad \Big|\, X^{-1}\cdot \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$X^{-1}AX = D$$

$$\boxed{X} \quad \boxed{D}$$

Are all Matrices Diagonalizable?

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}. \leftarrow \text{not diagonalizable}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \leftarrow$$

$$1x + 1y = y \Rightarrow y = 0$$

$$\cancel{0x + 1y = y}$$

$$\cancel{y = y}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ? \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Power Iteration

We can use linear-independence to find the eigenvector with the largest eigenvalue. Consider the eigenvalues of A^{1000}.

Power Iteration: Issues?

What could go wrong with Power Iteration?

- Overflow (un-normalised)
- $|\lambda_1| = |\lambda_2|$
- Starting vector has no comp. in $\tilde{x}_1$:
  not an issue due to roundg

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$Ax = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

$$\lambda x = \begin{pmatrix} \vdots \\ \vdots \end{pmatrix}$$

In principle: $A\vec{x} / \vec{x}$

$$\frac{\vec{x}\cdot A\vec{x}}{\vec{x}\cdot\vec{x}} = \text{Rayleigh quotient}$$

$$\frac{\vec{x}\cdot\lambda\vec{x}}{\vec{x}\cdot\vec{x}} = \lambda$$

Convergence of Power Iteration

What can you say about the convergence of the power method? Say $v_1^{(k)}$ is the kth estimate of the eigenvector x_1, and

$$e_k = \left\| x_1 - v_1^{(k)} \right\|.$$

$$\frac{A^{1000} x}{\lambda_1^{1000}} = \alpha_1 x_1 + \underbrace{\alpha_2 \frac{\lambda_2^{1000}}{\lambda_1^{1000}} x_2 + \cdots + \alpha_n \frac{\lambda_n^{1000}}{\lambda_1^{1000}} x_n}_{\text{error } e_k}$$

$$\|e_k\| \approx \left|\frac{\lambda_2}{\lambda_1}\right| \|e_{k-1}\|$$

$$|\lambda_1| > |\lambda_2|$$

Transforming Eigenvalue Problems

Suppose we know that $Ax = \lambda x$. What are the eigenvalues of these changed matrices?

Power. $A \to A^k$

$$A^2 \vec{x} = AA\vec{x} = A\lambda\vec{x} = \lambda^2\vec{x}$$

Shift. $A \to A - \sigma I$

$$(A - \sigma I)\vec{x} = A\vec{x} - \sigma I\vec{x} = \lambda\vec{x} - \sigma\vec{x} = (\lambda - \sigma)\vec{x}$$

Inversion. $A \to A^{-1}$

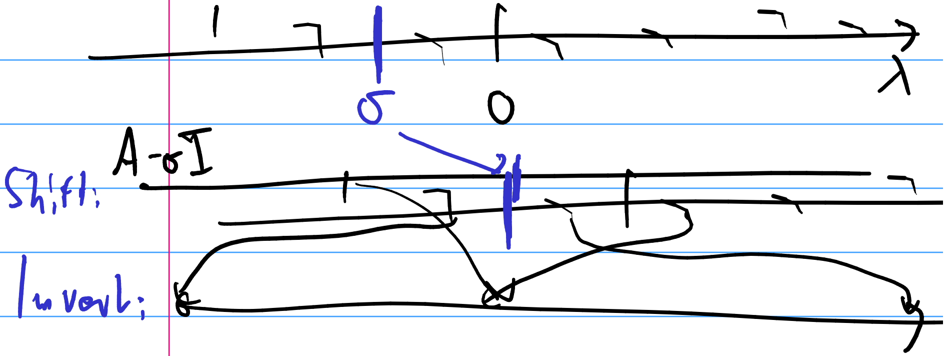
$$A^{-1}x = A^{-1}\frac{Ax}{\lambda} = \frac{x}{\lambda}$$

$$x = \frac{Ax}{\lambda}$$

λ	-5	5
$A - 3I$	$-8 = -5 - 3$	$2 = 5 - 3$
A	0.25	5
A^{-1}	$1/0.25 = 4$	0.2

A:	-3	0.2	5
$(A-2I)$:	-5	-2.2	3

A:	-3	0.2	1000
$A+2I$:	-1	2.2	1002
$(A+2I)^{-1}$:	-1	$\frac{1}{2.2}$	$\frac{1}{1002}$



"Inverse Iteration"

Inverse iteration + RQ as shift: RQI

Inverse Iteration / Rayleigh Quotient Iteration

Describe inverse iteration.

$$\vec{x}_{k+1} = (A - \sigma I)^{-1} \vec{x}_k \qquad \vec{x}_0 = ?$$

Describe Rayleigh Quotient Iteration.

$$\sigma = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}}$$

$$\vec{x}_{k+1} = (A - \sigma I)^{-1} \vec{x}_k$$

Demo: Power Iteration and its Variants

In-class activity: Eigenvalue Iterations

$$Ax = b$$

$$\Leftrightarrow x = A^{-1}b$$

la.solve( )

$$x_{k+1} = (A - \sigma I)^{-1} x_k$$

$$\Leftrightarrow (A - \sigma I) x_{k+1} = x_k$$

Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

$$x_{k+1} = A x \qquad A\underbrace{\begin{pmatrix} | & | \\ x & y \\ | & | \end{pmatrix}} = \begin{pmatrix} Ax & Ay \end{pmatrix}$$

orthogonalize

"~~simultaneous iteration~~"

"orthogonal iteration"

$$A\vec{x} = \lambda\vec{x}$$

$$\begin{pmatrix} \vec{x} & \underbrace{|\ \ |\ \ |\ \ |}_{\text{orth. to } \vec{x}} \end{pmatrix} A\, Q^{-1} = \begin{pmatrix} \lambda & ? \quad ? \\ | & \\ 0 & \boxed{\quad B \quad} \\ | & \end{pmatrix}$$

Q

"Deflation"

Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

$$Ax = \lambda x$$

$$y = \zeta x$$

$$Ay = \zeta A x = \zeta \lambda x$$

$$= \lambda y$$

Outline