<u>Overview</u>

SVD

↳ LSQ

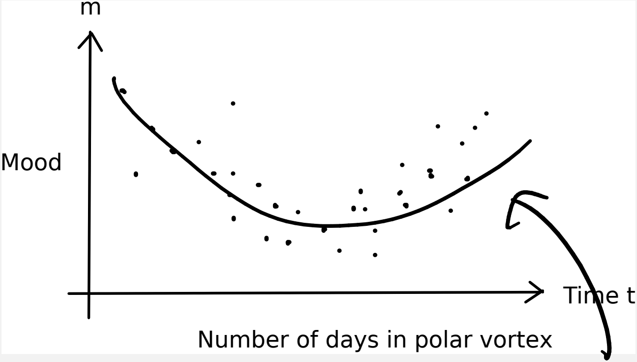
↳ $\|\cdot\|_2$

↳ LRA

Interpolation

Fitting a Model to Data

How can I fit a model to measurements? E.g.:



$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

find $\tilde{x}$ s.t. $\| \boxed{V} \tilde{\alpha} - \tilde{y} \|_2$ is minimal

$$V\vec{\alpha} \cong \vec{b}$$

"least-squares equal"

i.e. minimize the 2-norm of the residual

Demo: Data Fitting using Least Squares

Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^T$$

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$= m \quad \|U \Sigma V^T x\|_2$$

$$= \max \| \Sigma V^T x \|_2$$

$\rightarrow \|x\|_2 = 1$

same

$$= \max_{\|V^T x\|_2 = 1} \|\Sigma V^T x\|_2$$

V orth. $\Rightarrow \|x\|_2 = \|V^T x\|_2$

$$= \max_{\|y\|_2 = 1} \|\Sigma y\|_2$$

$$= \|\Sigma\|_2$$

$$= \sigma_1$$

$$\rightarrow \|Qx\|_2^2 = (Qx) \cdot (Qx) = x^T Q^T Q x = \|x\|_2^2$$

Condition Numbers

How would you compute a 2-norm condition number?

$$\operatorname{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$= \sigma_1 \cdot \frac{1}{\sigma_n}$$

$$A^{-1} = V \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} U^T$$

Outline

SVD as Sum of Outer Products

What's another way of writing the SVD?

$$A = U \Sigma V^T = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix} \begin{pmatrix} - & \sigma_1 v_1 & - \\ & \vdots & \\ - & \sigma_n v_n & - \end{pmatrix}$$

$$= \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

$$A_3 = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T$$

$$3(m+n)+3$$

$$A_3 \approx A$$

10x10 images of faces

$$\vec{f}_1, \ldots, \vec{f}_{5000}$$

$$F = \begin{array}{c c} & 5000 \\ 100 & \begin{pmatrix} | & & \\ \vec{f}_1 & \cdots & \vec{f}_{5000} \\ | & & \end{pmatrix} \end{array}$$

$\sigma_1 \ldots \sigma_{10}$ big

$$\approx \begin{array}{c c} & 100 \\ 100 & \begin{pmatrix} \uparrow & & & \\ & \downarrow & & \\ & & U & \end{pmatrix} \end{array} \begin{array}{c c} & 100 \\ 100 & \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{100} \end{pmatrix} \end{array} \begin{array}{c c} & 5000 \\ 100 & \begin{pmatrix} V \end{pmatrix} \end{array}$$

Low-Rank Approximation (I)

What is the *rank* of $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T$?

What is the *rank* of $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T$?

Demo: Image Compression

Low-Rank Approximation

What can we say about the low-rank approximation

$$A_k = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T \cancel{+ \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^T}$$

to

$$A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + \sigma_1 \boldsymbol{u}_n \boldsymbol{v}_n^T ?$$

For simplicity, assume $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$.

$$\min_{\operatorname{rank} B = k} \|A - B\|_2 = \|A - A_k\|_2$$

(Eckart-Young-Mirski theorem)

$$\min_{\operatorname{rank} B = k} \|A - B\|_F = \|A - A_k\|_F$$

Outline