

Overview

SVD

↳ LSA

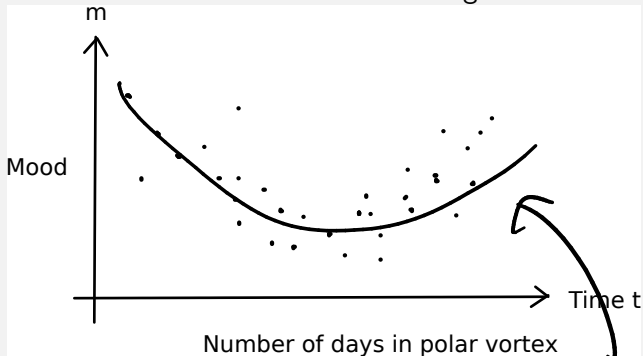
↳ $\| \cdot \|_2$

↳ CRA

Interpolation

Fitting a Model to Data

How can I fit a model to measurements? E.g.:



$$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$$

Find $\vec{\alpha}$ s.t. $\| \vec{\alpha} - \vec{y} \|_2$ is minimized

$$V \approx \underset{\approx}{=} b$$

↑
"least-squares
equal"

i.e., minimize the L_2 -norm
of the residual

Demo: Data Fitting using Least Squares

Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^T$$

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

$$= \max \|U \Sigma V^T x\|_2$$

$$= \max \| \Sigma V^T x \|_2$$

$$\rightarrow \|x\|_2 = 1$$

same

$$= \max_{\|V^T x\|_2 = 1} \|\Sigma V^T x\|_2$$

$$= \max_{\|y\|_2 = 1} \|\Sigma y\|_2$$

$$= \|\Sigma\|_2$$

$$= \sigma_1$$

Vorh.

$$\Rightarrow \|x\|_2 = \|V^T x\|_2$$

$$\rightarrow \|Qx\|_2^2 = (Qx) \cdot (Qx) = x^T Q^T Q x = \|x\|_2^2$$

Condition Numbers

How would you compute a 2-norm condition number?

$$\text{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$$

$$= \sigma_1 \Rightarrow \frac{1}{\sigma_n}$$

$$A^{-1} = V \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} W^T$$

↑

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
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Floating Point
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 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
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Approximate Undo: SVD and Least Squares

SVD: Applications

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Iteration and Convergence

Solving One Equation

Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

SVD as Sum of Outer Products

What's another way of writing the SVD?

$$\begin{aligned} A = U \Sigma V^T &= \begin{pmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_n & \\ & & & 0 \end{pmatrix} \begin{pmatrix} - & v_1^T \\ & \vdots \\ - & v_n^T \end{pmatrix} \\ &= \begin{pmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{pmatrix} \begin{pmatrix} - & \sigma_1 & v_1^T \\ & \vdots & \\ - & \sigma_n & v_n^T \end{pmatrix} \\ &= \underbrace{\sigma_1 u_1 v_1^T}_{\text{outer product}} + \sigma_2 u_2 v_2^T + \dots + \sigma_n u_n v_n^T \end{aligned}$$

$$A = \underbrace{\sigma_1 u_1 v_1^T} + \sigma_2 u_2 v_2^T + \dots + \sigma_n u_n v_n^T$$

$$A_3 = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T$$

↑
3 (n+1) + 3

$$A_3 \approx A$$

10x10 images
of faces

$$f_1 \dots f_{5000}$$

$$F = 100 \begin{pmatrix} f_1 & \dots & f_{5000} \end{pmatrix} \quad \sigma_1 \dots \sigma_n \text{ big}$$

$$\approx \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ 100 & & & \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \sigma_1 & \dots & \sigma_n & \\ \downarrow & & & \\ 100 & & & \end{pmatrix} \begin{pmatrix} V \\ \vdots \\ \end{pmatrix}$$

5000

Low-Rank Approximation (I)

What is the *rank* of $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T$?

What is the *rank* of $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T$?

Demo: Image Compression

Low-Rank Approximation

What can we say about the low-rank approximation

$$A_k = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T \quad \left(\text{---} + \sigma_k \mathbf{u}_k \mathbf{v}_k^T \text{---} \right)$$

to

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T? \quad \leftarrow$$

For simplicity, assume $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$.

$$\min_{\text{rank } B = k} \|A - B\|_2 = \|A - A_k\|_2$$

(Eckart-Young-Mirski theorem)

$$\min_{\text{rank } B = k} \|A - B\|_F = \|A - A_k\|_F$$

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