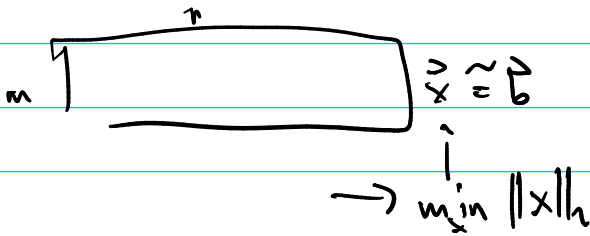
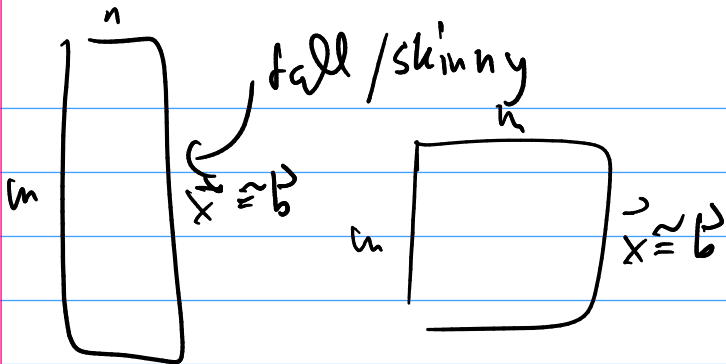


Overview

- Calc. on
Interpolants
 - ↳ Shortcuts
 - ↳ Accuracy
- Iterative
processes



$$\sum \vec{y} \approx U^T b$$

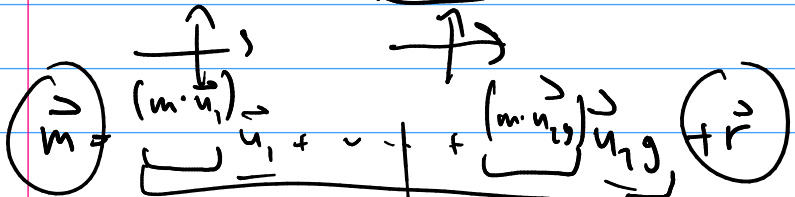
$$\vec{x} \cdot \vec{y} = 0$$

$$\Rightarrow Q_x^T \cdot Q_y = 0$$

$$U \subseteq V$$



=



Calculus on Interpolants

Suppose we have an interpolant $\tilde{f}(x)$ with $f(x_i) = \tilde{f}(x_i)$ for $i = 1, \dots, n$:

$$\hookrightarrow \tilde{f}(x) = \alpha_1 \varphi_1(x) + \dots + \alpha_n \varphi_n(x)$$

How do we compute the derivative of \tilde{f} ?

$f'(x_i)$?
can't find

$$f'(x) \approx \tilde{f}'(x) = \alpha_1 \varphi_1'(x) + \dots + \alpha_n \varphi_n'(x)$$

Suppose we have function values at nodes $(x_i, f(x_i))$ for $i = 1, \dots, n$ for a function f . If we want $f'(x_i)$, what can we do?

$\alpha_1, \dots, \alpha_n \leftarrow$ have

$$\tilde{f}'(x_i) = \alpha_1 \varphi_1'(x_i) + \dots + \alpha_n \varphi_n'(x_i)$$

$$\begin{pmatrix} \psi_1'(x_1) \\ \vdots \\ \psi_n'(x_n) \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \tilde{\psi}_1'(x_1) \\ \vdots \\ \tilde{\psi}_n'(x_n) \end{pmatrix}$$

$$V^{-1} \alpha^0 = \beta^0$$

$$V \alpha^0 = \beta^0 \Leftrightarrow \alpha^0 = V^{-1} \beta^0$$

$$\alpha^0 = (V^{-1} \beta^0)$$

$D \approx V' V^{-1}$ \leftarrow differentiation
mat

h^2 \leftarrow take two
derivatives

$V' V^{-1}$

f

$$|f - \tilde{f}| \leq C \cdot h^n$$

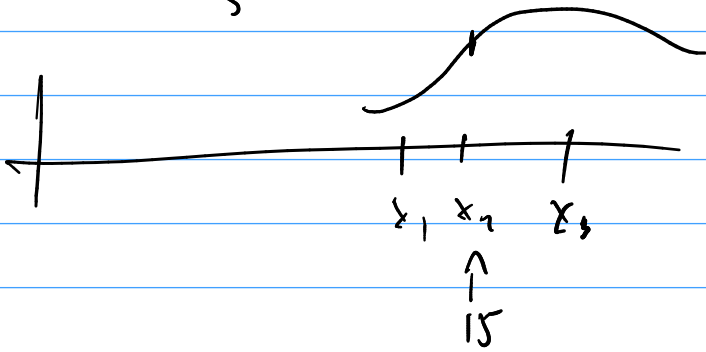
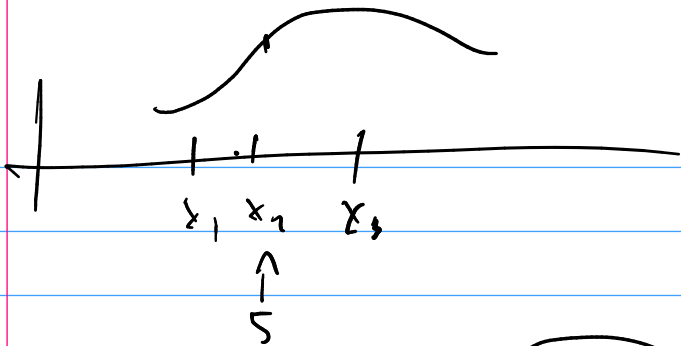
Interpolation

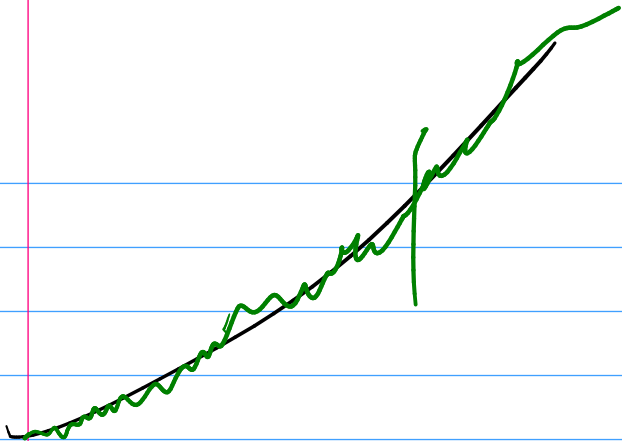
$$|f' - \hat{f}'| \leq C \cdot h^{n-1}$$

↑
maybe h^2 ?

$\lim_{h \rightarrow 0}$

$$\frac{f(x+h) - f(x)}{h}$$





$$f'(x) \approx \frac{1}{4} \left[f\left(x + \frac{1}{8}\right) - f\left(x - \frac{1}{8}\right) \right]$$

$$\rightarrow \frac{1}{h} \left[f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \right]$$

About Differentiation Matrices

How could you find coefficients of the derivative in the original basis (φ_i) ?

Give a matrix that finds the second derivative.

Demo: Taking derivatives with Vandermonde matrices

Finite Difference Formulas

It is possible to use the process above to find 'canned' formulas for taking derivatives. Suppose we use three points equispaced points $(x - h, x, x + h)$ for interpolation (i.e. a degree-2 polynomial).

- ▶ What is the resulting differentiation matrix?
- ▶ What does it tell us for the middle point?

Can we use a similar process to compute (approximate) integrals of a function f ?

Example: Building a Quadrature Rule

Demo: Computing the Weights in Simpson's Rule

Suppose we know

$$f(x_0) = 2 \quad f(x_1) = 0 \quad f(x_2) = 3$$

$$x_0 = 0 \quad x_1 = \frac{1}{2} \quad x_2 = 1$$

How can we find an approximate integral?

$$\tilde{f}(x) = \alpha_0 p_0(x) + \alpha_1 p_1(x) + \alpha_2 p_2(x)$$

$$\int f(x) dx \approx \int_0^1 \tilde{f}(x) dx = \alpha_0 \int_0^1 p_0(x) dx + \alpha_1 \int_0^1 p_1(x) dx + \alpha_2 \int_0^1 p_2(x) dx$$

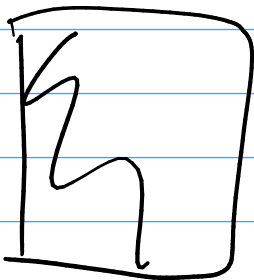
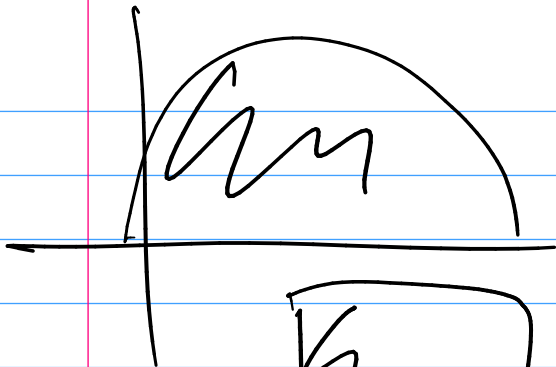
$$\int \tilde{p}(x) = j \cdot Q^N \quad j = \begin{pmatrix} \int +_1(x) dx \\ i \\ \int p_3(x) dx \end{pmatrix}$$

$$= j \cdot V^{-1} \cdot \frac{Q^N}{\partial V}$$

$$= \left(j \cdot V^{-1} \right) \cdot \frac{Q^N}{\partial V}$$

$$= \frac{Q^N}{\partial V}$$

$$\frac{Q^N}{\partial V} = j \cdot V^{-1}$$



← Quadrat

$$j_1 = \int_0^1 1 \, dx = 1$$

$$j_2 = \int_0^1 x \, dx = \frac{1}{2}$$

$$j_3 = \int_0^1 x^2 \, dx = \frac{1}{3}$$

Facts about Quadrature

What does Simpson's rule look like on $[0, 1/2]$?

What does Simpson's rule look like on $[5, 6]$?

How accurate is Simpson's rule with n points and functions?

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems

Data Fitting

Norms and Condition Numbers

Low-Rank Approximation

Interpolation

Iteration and Convergence

Solving One Equation

Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

What is linear convergence? quadratic convergence?