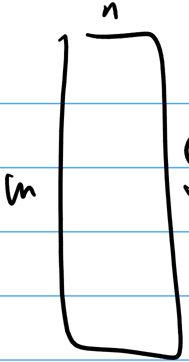
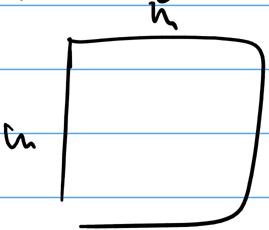
Overview

- Calc. on Interpolants
  - ↳ Shortcuts
  - ↳ Accuracy
- Iterative processes

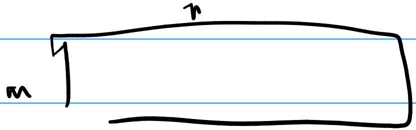


tall/skinny

$\vec{x} \approx \vec{b}$



$\vec{x} \approx \vec{b}$



$\vec{x} \approx \vec{b}$

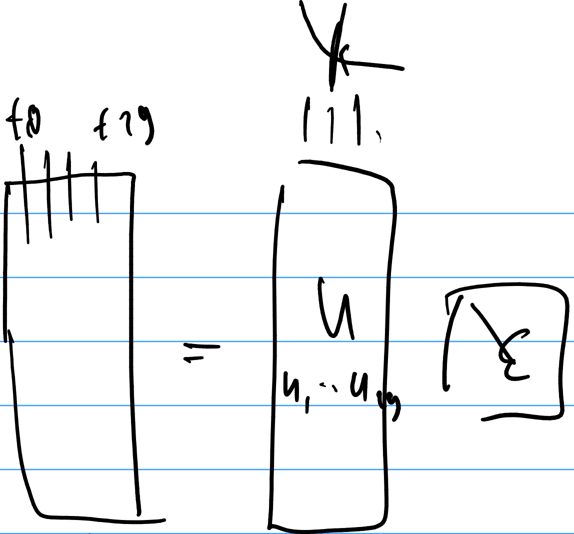
$\longrightarrow \min_x \|x\|_2$

$$\Sigma \vec{y} = U' b$$

$$\vec{x} \cdot \vec{y} = 0$$

$$\Rightarrow Q\vec{x} \cdot Qy = 0$$

$$U \subset V$$



$$\vec{m} = \underbrace{(m \cdot \vec{u}_1)}\,\vec{u}_1 + \cdots + \underbrace{(m \cdot \vec{u}_{29})}\,\vec{u}_{29} + \vec{r}$$

Calculus on Interpolants

Suppose we have an interpolant $\tilde{f}(x)$ with $\underset{\sim}{f(x_i)} = \tilde{f}(x_i)$ for $i = 1, \ldots, n$:

$$\hookrightarrow \quad \tilde{f}(x) = \alpha_1 \varphi_1(x) + \cdots + \alpha_n \varphi_n(x)$$

How do we compute the derivative of $\tilde{f}$?

$f'(x_i)$?

can't find

$$f'(x) \approx \tilde{f}'(x) = \alpha_1 \varphi_1'(x) + \cdots + \alpha_n \varphi_n'(x)$$

Suppose we have function values at nodes $(x_i, f(x_i))$ for $i = 1, \ldots, n$ for a function f. If we want $f'(x_i)$, what can we do?

$\alpha_1 \ldots \alpha_n \leftarrow$ have

$$\tilde{f}'(x_1) = \alpha_1 \varphi_1'(x_1) + \cdots + \alpha_n \varphi_n'(x_1)$$

$$\underbrace{\begin{pmatrix} \varphi_1'(x_1) & \cdots & \varphi_n'(x_1) \\ \vdots & & \vdots \\ \varphi_1'(x_n) & \cdots & \varphi_n'(x_n) \end{pmatrix}}_{V'} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \tilde{f}'(x_1) \\ \vdots \\ \tilde{f}'(x_n) \end{pmatrix}$$

$$V' \tilde{\alpha} = \tilde{f}'$$

$$V \tilde{\alpha} = \tilde{f} \iff \tilde{\alpha} = V^{-1} \tilde{f}$$

$$\tilde{f}' = (V' V^{-1}) \tilde{f}$$

2

$D = V' \; V^{-1}$ ⟸ differentiation mat

D^2 ⟸ take two derivatives

$V'V^{-1}$

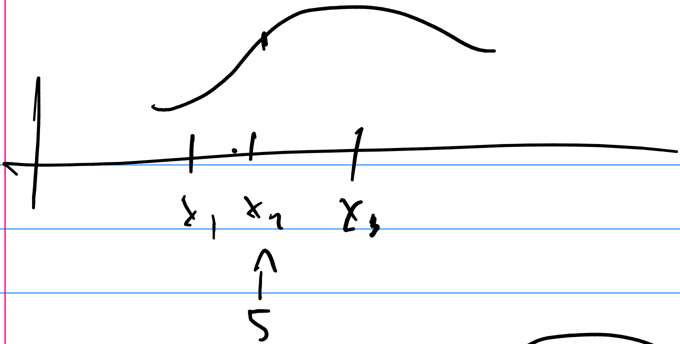
f $\quad |f - \tilde{f}| \leq C \cdot h^n$

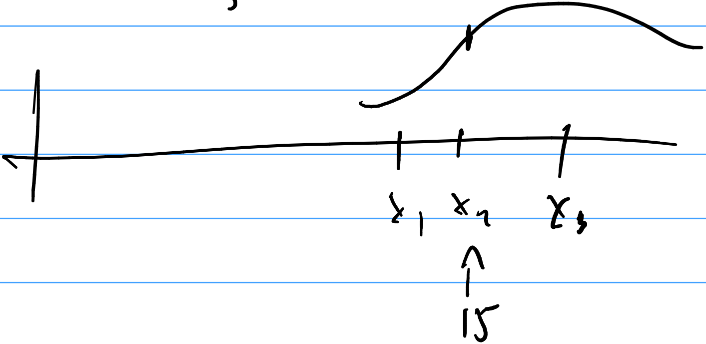
Interpolation

$$|f' - \hat{f}'| \leq C \cdot h^{n-1}$$

maybe n?

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$


x_1 x_2 x_3
5


x_1 x_2 x_3
15

$$f'(x) \approx \frac{1}{\frac{1}{4}}\left[f\left(x+\frac{1}{8}\right)-f\left(x-\frac{1}{8}\right)\right]$$

$$\longrightarrow = \frac{1}{h}\left[f\left(x+\frac{h}{2}\right)-f\left(x-\frac{h}{2}\right)\right]$$

About Differentiation Matrices

How could you find coefficients of the derivative in the original basis (φ_i)?

Give a matrix that finds the second derivative.

Demo: Taking derivatives with Vandermonde matrices

Finite Difference Formulas

It is possible to use the process above to find 'canned' formulas for taking derivatives. Suppose we use three points equispaced points $(x-h, x, x+h)$ for interpolation (i.e. a degree-2 polynomial).

- What is the resulting differentiation matrix?
- What does it tell us for the middle point?

Can we use a similar process to compute (approximate) integrals of a function f?

Example: Building a Quadrature Rule

Demo: Computing the Weights in Simpson's Rule

Suppose we know

$$f(x_0) = 2 \qquad f(x_1) = 0 \qquad f(x_2) = 3$$

$$x_0 = 0 \qquad x_1 = \frac{1}{2} \qquad x_2 = 1$$

How can we find an approximate integral?

$$\tilde{f}(x) = \alpha_1 \varphi_1(x) + \alpha_2 \varphi_2(x) + \alpha_3 \varphi_3(x)$$

$$\int f(x)\,dx \approx \int_0^1 \tilde{f}(x)\,dx = \alpha_1 \int \varphi_1(x)\,dx + \alpha_2 \int \varphi_2(x)\,dx + \alpha_3 \int \varphi_3(x)\,dx$$

$$\int \hat{f}(x) = \vec{j} \cdot \vec{\alpha} \qquad \vec{j} = \begin{pmatrix} \int \varphi_1(x)\,dx \\ \vdots \\ \int \varphi_3(x)\,dx \end{pmatrix}$$

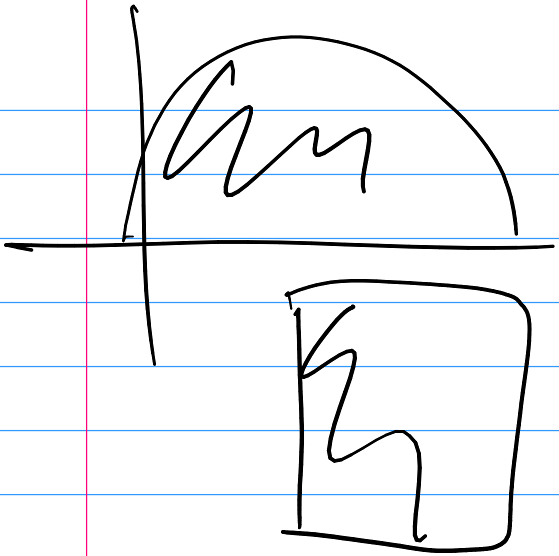
$$\vec{\alpha} = V^{-1} \vec{f}$$

$$= \vec{j} \cdot V^{-1} \vec{f}$$

$$= \left(\vec{j}^T V^{-1} \right) \cdot \vec{f}$$

$$= \vec{w} \cdot \vec{f}$$

$$\vec{w} = \vec{j}^T V^{-1}$$



← Quadrat

$$j_1 = \int_0^1 1 \, dx = 1$$

$$j_2 = \int_0^1 x \, dx = \frac{1}{2}$$

$$j_3 = \int_0^1 x^2 \, dx = \frac{1}{3}$$

Facts about Quadrature

What does Simpson's rule look like on $[0, 1/2]$?

What does Simpson's rule look like on $[5, 6]$?

How accurate is Simpson's rule with n points and functions?

Outline

What is linear convergence? quadratic convergence?