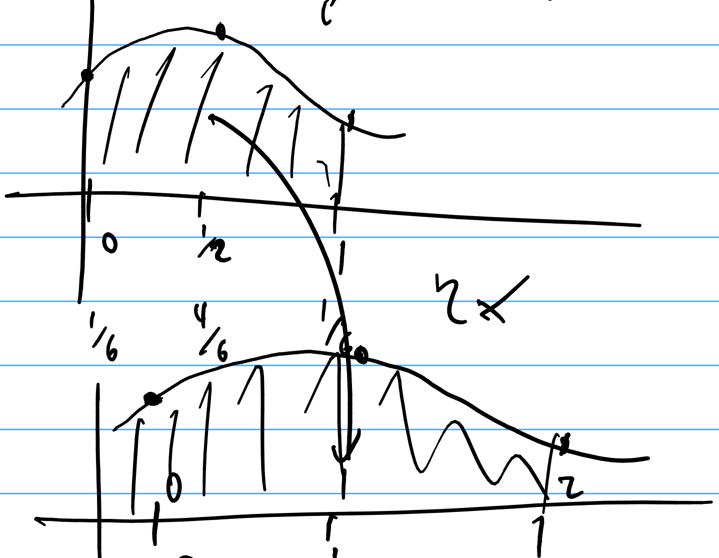
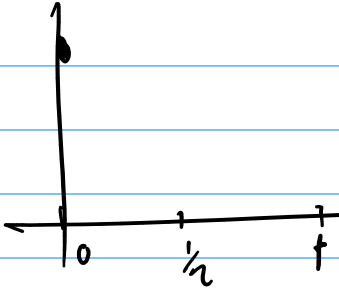
Overview

- Shifting / scaling
- Acc quad
- Convergance
- Solving eq.

Simpsons rule





$$\int_0^1 f \simeq \frac{1}{6} f(0) + \frac{4}{6} f(1/2) + \frac{1}{6} f(1)$$

$\downarrow$

$$\int_0^2 f \simeq \frac{2}{6} f(0) + \frac{4}{6} f(1) + \frac{2}{6} f(2)$$

To find a quadrature rule on any interval $[a,b]$:

1. Cook up $\varphi(x) = mx + n$ so that $\varphi(0) = a$ $\varphi(1) = b$
2. new nodes: $\tilde{x}_i = \varphi(x_i)$
3. weights: $\tilde{w}_i = \varphi'(x_i)\, w_i$

$$= m \cdot w_i$$

Example: Building a Quadrature Rule

Demo: Computing the Weights in Simpson's Rule

Suppose we know

$$f(x_0) = 2 \qquad f(x_1) = 0 \qquad f(x_2) = 3$$

$$x_0 = \cancel{1}\ 0 \qquad x_1 = \frac{1}{2} \qquad x_2 = 1$$

How can we find an approximate integral?

Facts about Quadrature

nodes: $0, \frac{1}{2}, 1$ w: $\frac{1}{6}, \frac{4}{6}, \frac{1}{6}$

What does Simpson's rule look like on $[0, 1/2]$?

$$\underbrace{\frac{1}{12}} f(0) + \underbrace{\frac{4}{6}\cdot\frac{1}{2}} f\left(\frac{1}{4}\right) + \underbrace{\frac{1}{12}} f\left(\frac{1}{2}\right)$$

What does Simpson's rule look like on $[5, 6]$?

$$\underbrace{\frac{1}{6}} f(5) + \underbrace{\frac{4}{6}} f(5.5) + \underbrace{\frac{1}{6}} f(6)$$

How accurate is Simpson's rule with polynomials of degree n?

$$\int_0^h = h \cdot \int_0^1 \text{interpolant}$$

Error for interpolation

$$|f(x) - \tilde{f}(x)| \le \underline{C \cdot h^{n+1}}$$

n poly degree

$$\left|\int f - \int \tilde{f}\right| \le \int_0^h |f - \tilde{f}(x)|$$

$$\le C \cdot \int_0^h C \cdot h^{n+1}$$

$$= C \cdot \underline{h} \int_0^1 h^{n+1}$$

$$\leq C \cdot h^{n+2} \int_0^1 \cdots$$


h
h/2

Why do the weights in Simpson's rule add up to 1?

$$\underbrace{\frac{1}{6}}\cdot f(0) + \underbrace{\frac{4}{6}}\, f\left(\frac{1}{2}\right) + \underbrace{\frac{1}{6}}\cdot f(1)$$

$$= \frac{1}{6} + \frac{4}{6} + \frac{1}{6}$$

$$v'\left(v^{-1}\left(\vec{p}\,\right)\right)$$

$$\text{eval\_deriv.}\left(\text{compute\_coeff}\left(\vec{p}\,\right)\right)$$

$$\begin{pmatrix} f'(0) \\ f'(1) \\ f'(2) \\ f'(3) \\ f'(4) \end{pmatrix} = \underbrace{\begin{bmatrix} & & & & \\ \circ & \circ & \circ & \circ & \circ \\ & & & & \end{bmatrix}}_{V'V^{-1}} \begin{pmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \end{pmatrix}$$

$$f'(2) = \underline{\qquad}\, f(0) + \dots + \underline{\qquad}\, f(4)$$

Outline

What is linear convergence? quadratic convergence?

$$e_k = \| x_k - \hat{x} \|$$

$$e_{k+1} = \underbrace{\frac{\lambda_l}{\lambda_1}}_{e_k} \cdot e_k$$

"linear convergence"

$$\frac{e_{k+1}}{e_k} = C \qquad e_{k+1} = C \cdot e_k$$

actually works if $C < 1$

"quadratically convergent"

$\frac{e_{k+1}}{e_k^2} = C \iff e_{k+1} = \underline{C} \cdot e_k^2$

$e_1 = 0.1 \qquad C = 0.9$

About Convergence Rates

Demo: Rates of Convergence

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

linear: gains a const. # of digits

quadr: doubles # of digits

Outline

Solving Nonlinear Equations

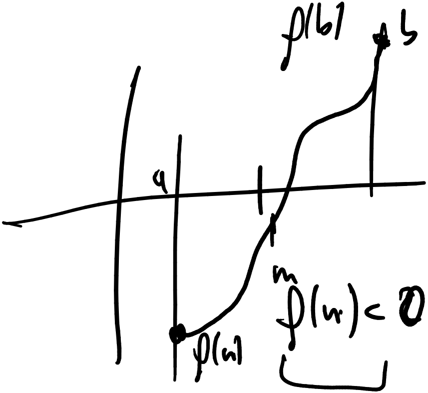
What is the goal here?

$$\underbrace{x^{17} + 3x^3 + 5x^2 + 3x + 7}_{f(x)} = 0$$

$f(a) < 0$

$f(b) > 0$

Bisection



$$\underset{64}{\underset{\updownarrow}{\tilde{a}}} + \underset{64}{\underset{\updownarrow}{\tilde{b}}} \cdot 2^{-57}$$

$$\left(a + b\, 2^{-57}\right) \cdot \left(c + d \cdot 2^{-57}\right)$$

Bisection Method

Assume continuos function f has a zero on the interval $[a, b]$ and

$$\text{sign}(f(a)) = -\text{sign}(f(b)).$$

Perform binary search: check sign of $f((a + b)/2)$ and define new search interval so that ends have opposite sign.

Demo: Bisection Method

What's the rate of convergence? What's the constant?

Newton's Method

Derive Newton's method.