Taylor Remainders: the Full Truth

Let $f : \mathbb{R} \to \mathbb{R}$ be $(n+1)$-times differentiable on the interval (x_0, x) with $f^{(n)}$ continuous on $[x_0, x]$. Then there exists a $\xi \in (x_0, x)$ so that

$$f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i = \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}}_{\text{“}C\text{”}} \cdot (\xi - x_0)^{n+1}$$

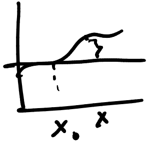
and since $|\xi - x_0| \leqslant h$

$$\left| f(x_0 + h) - \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} h^i \right| \leqslant \underbrace{\frac{\left| f^{(n+1)}(\xi) \right|}{(n+1)!}}_{\text{“}C\text{”}} \cdot h^{n+1}.$$

Intuition for Taylor Remainder Theorem

Given the value of a function and its derivative $f(x_0), f'(x_0)$, prove the Taylor error bound.

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(w_0)\, dw_0$$



$$f'(w_0) = f'(x_0) + \int_{x_0}^{w_0} f''(w_1)\, dw_1$$

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(x_0) + \int_{x_0}^{w_0} f''(w_1)\, dw_1\, dw_0$$

$$= \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{\text{Taylor } t_1(x)} + \int_{x_0}^{x} \int_{x_0}^{w_0} f''(w_1)\, dw_1\, dw_0$$

$$\xi = \operatorname{argmax}_{\xi \in [x_0, x]} |f''(\xi)|$$

$$|f(x) - t_1(x)| = | \quad \leq \int_{x_0}^{x} \int_{x_0}^{w_0} |f''(\xi)|\, dw_1\, dw_0$$

In-class activity: Taylor series

$$|f(x) - t_1(x)| \leq \int_{x_0}^{x} \int_{x_0}^{w_0} |f''(\xi)| \, dw_1 \, dw_0$$

$$= \int_{x_0}^{x} |f''(\xi)| \, |(w_0 - x_0)| \, dw_0$$

$$= \frac{|f''(\xi)| \, |x - x_0|^2}{2!}$$

$$|f(x_0+h) - t_1(x_0+h)| \leq \frac{|f''(\xi)|}{2!} h^2 = O(h^2)$$

Proof of Taylor Remainder Theorem

We can complete the proof by induction of the Taylor degree expansion n

$$f(x) = t_{n-1}(x) + \int_{x_0}^{x} \int_{x_0}^{w_0} \cdots \int_{x_0}^{w_{n-1}} f^{(n)}(w_n) dw_n \cdots dw_0$$

Using Polynomial Approximation

Suppose we can approximate a function as a polynomial:

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

How is that useful?
E.g.: What if we want the integral of f?

Demo: Computing π with Taylor

Reconstructing a Function From Point Values

If we know function values at some points $f(x_1), f(x_2), \ldots, f(x_n)$, can we reconstruct the function as a polynomial?

$$f(x) = \underset{a_0}{???} + \underset{a_1}{???}x + \underset{a_2}{???}x^2 + \cdots$$

$$a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 + \ldots = f(x_1)$$

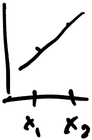
$$\vdots$$

$$a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \ldots a_{n-1} x_n^{n-1} = f(x_n)$$

$n=2$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & x_n^2 & & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$



Vandermonde

Vandermonde Linear Systems

Polynomial interpolation is a critical component in many numerical models.

monomials: $x^0, x^1, x^2, \dots x^{n-1}$

$\varphi_0(x), \varphi_1(x), \dots \varphi_{n-1}(x)$

$$\begin{pmatrix} \varphi_0(x_0) & \cdots & \varphi_{n-1}(x_0) \\ & \vdots & \\ \varphi_0(x_{n-1}) & \cdots & \varphi_{n-1}(x_{n-1}) \end{pmatrix} \begin{pmatrix} ? \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_{n-1}) \end{pmatrix}$$

Generalized Vandermonde

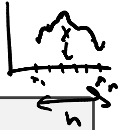
Demo: Polynomial Approximation with Point Values

Error in Interpolation

How did the interpolation error behave in the demo?

To fix notation: f is the function we're interpolating. $\tilde{f}$ is the interpolant that obeys $\tilde{f}(x_i) = f(x_i)$ for $x_i = x_1 < \ldots < x_n$. $h = x_n - x_1$ is the interval length.

$$|f(x) - \tilde{f}(x)| = O(h^{n+1})$$



What is the error *at* the interpolation nodes?

0

Care to make an unfounded prediction? What will you call it?

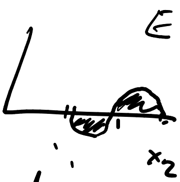
nth order convergence

Proof Intuition for Interpolation Error Bound

Let us consider an interpolant $\tilde{f}$ based on $n = 2$ points so

$$\tilde{f}(x_1) = f(x_1) \quad \text{and} \quad \tilde{f}(x_2) = f(x_2).$$

Prove the interpolation error bound in this case.



$$E(x) = \tilde{f}(x) - f(x)$$

$$E(x_1) = E(x_2) = 0$$

$$E(x) = \underbrace{E(x_1)}_{=0} + \int_{x_1}^{x} E'(w_0)\, dw_0$$

$$E(x_2) - E(x_1) = \int_{x_1}^{x_2} E'(w_0)\, dw_0 = 0$$