Matrix Norms

What norms would we apply to matrices?

basis $U = (\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3)$

basis $V = (\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3)$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$y = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

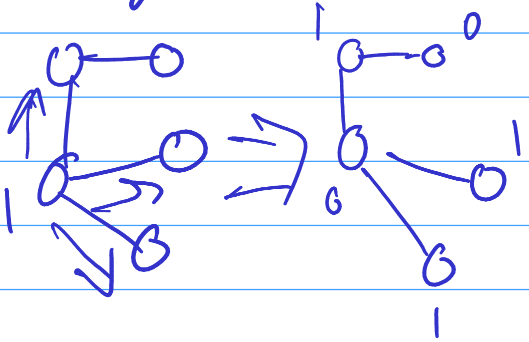
Alternative

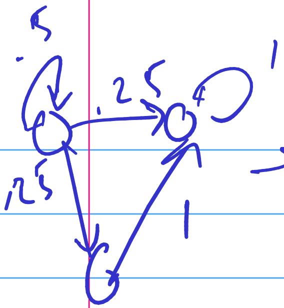
$y = Ax$ ← $f(x)$

$$y_i = \sum_j a_{ij} x_j$$

Matrix-vector mult. with adjacency matrix A

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$





$\to A$

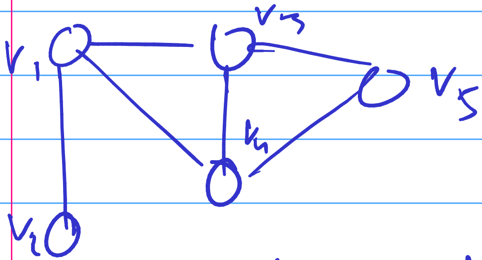
k-steps   1 step

$$A^k = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \leftarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$A^k x = \underbrace{A \cdots A \cdot}_{k\text{-times}} \overbrace{(A x)}$$

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Laplacians matrix

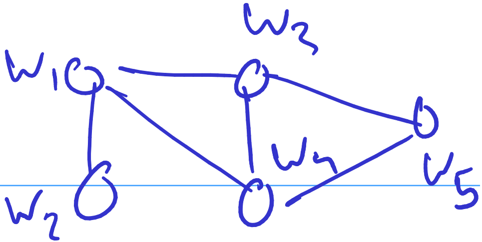


adjacency matrix A

degrees D

$L = D - A$

$$w = L \cdot v$$



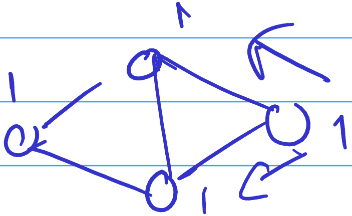
$$w = Dv - Av$$

$$w_i = \underbrace{Dv_i}_{0} - \sum_j A_{ij} v_j$$

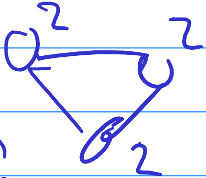
↑ 1 if j is con. i

$$\underbrace{w_i}_{0} = Dv_i - \sum_j A_{ij} v_j$$

If $Lv = 0$



$3v_2 = v_1 + v_3 + v_4$



$0 = (v_1 - v_2) + (v_3 - v_2)$
$+ (v_4 - v_2)$

Demo: Matrix norms

In-class activity: Matrix norms

induced "p-norms"

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$$

$$= \max_{x} \frac{\|Ax\|_2}{\|x\|_2}$$

Aside: if A is sym.

maximizer x of $\|Ax\|_2$

will be the eigenvector

with the largest eigenvalue

$\|A\|_2$ = largest eig-value of A
in abs.

Properties of Matrix Norms

Matrix norms inherit the vector norm properties:

1. $\|A\| > 0 \Leftrightarrow A \neq \mathbf{0}$. nonnegative and zero only if $A=0$
2. $\|\gamma A\| = |\gamma| \, \|A\|$ for all scalars γ.
3. Obeys triangle inequality $\|A + B\| \leqslant \|A\| + \|B\|$

But also some more properties that stem from our definition: of p-norm

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

submultiplicativity

$$\|AB\|_p \leq \|A\|_p \|B\|_p$$

$$\|Ax\|_F \leq \|A\|_F \|x\|_F$$

$$\left\| \begin{bmatrix} \rho_{11} & & \\ & \rho_{22} & \\ & & \rho_{33} \end{bmatrix} \right\|_2$$

$$\max_{\|x\|=1} \left\| \begin{bmatrix} \rho_{11} & & \\ & \rho_{22} & \\ & & \rho_{33} \end{bmatrix} x \right\|_2$$

$$\max_{\|x\|=1} \left\| \begin{matrix} D_{11} x_1 \\ D_{22} x_2 \\ D_{33} x_3 \end{matrix} \right\|_2$$

$$\max_{\|x\|=1} \sqrt{(D_{11} x_1)^2 + (D_{22} x_2)^2 + (D_{33} x_3)^2}$$

$$\sqrt{\left(\max_i D_{ii} \cdot 1\right)^2} = \max_i D_{ii}$$

$$\left\| \begin{bmatrix} D_{11} & & \\ & D_{22} & \\ & & D_{33} \end{bmatrix} \right\|_\infty = \left\| \begin{matrix} D_{11} \\ D_{22} \\ D_{33} \end{matrix} \right\|_\infty$$

$$= \max |D_{ii}|$$

max

$$\|x\|_\infty = 1 \qquad x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\|A\|_\infty = \max_i \sum_j |A_{ij}|$$

$$\|A\|_1 = \max_{\|x\|_1 = 1} Ax$$

if $x_i = \alpha$ $\quad \|A\|_1 \neq \sum_j \alpha \cdot A_{ij}$

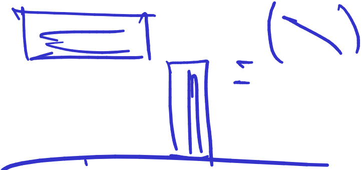
$\sum_i |x_i| = 1$ $\quad x = \begin{pmatrix} 0 \\ 0 \\ \neq 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \|A\|_1 = \max_j \sum_i |A_{ij}|$

Example: Orthogonal Matrices

What is the 2-norm of an orthogonal matrix?

Q is orthogonal

$$Q^T Q = I$$



$$\max_{\|x\|_1} \|Qx\|_2 = \max_{\|x\|=1} \sqrt{(Qx)^T \cdot Qx}$$

$$\|y\|_2^2 = y^T y$$

$$\|Q\|_2 = \max_{\|x\|_2=1} \sqrt{(Qx)^T \cdot Qx}$$

$$= \max_{\|x\|_2=1} \sqrt{x^T \underbrace{Q^T Q}_{I} x}$$

$$= \max_{\|x\|_2=1} \sqrt{x^T x} = 1$$

$$\|y\|_2^2 = y^T y$$

$$= \sum_i |y|^2$$

$$\|y\|_2 = \sqrt{y^T y}$$

Conditioning

Now, let's study condition number of solving a linear system

$$Ax = b.$$

x: ? given b

$A(x + \Delta x) = b + \Delta b$

output error (Δx) input error (Δb)

rel error $= \dfrac{\|\Delta x\| \, \|b\|}{\|\Delta b\| \, \|x\|}$

$$\text{rel error} = \frac{\text{rel error in out.}}{\text{rel error in input}}$$

$$= \frac{\|\Delta x\| / \|x\|}{\|\Delta b\| / \|b\|}$$

rel err $= \dfrac{\|Ax\| \, \|b\|}{\|\Delta b\| \, \|x\|}$

$= \dfrac{\|A^{-1}\Delta b\| \, \|Ax\|}{\|\Delta b\| \, \|x\|}$

$\leq \|A^{-1}\| \cdot \|A\| = \underbrace{\dfrac{\|\Delta b\| \|x\|}{\|\Delta b\| \|x\|}}_{1}$

$A^{-1}Ab \qquad Ax$

$$\text{rel err} = \frac{\|Ax\|\,\|b\|}{\|Ab\|\,\|x\|}$$

$$= \frac{\|A^{-1}\cdot Ab\|\cdot\|Ax\|}{\|Ab\|\,\|x\|}$$

$$\le \underbrace{\|A^{-1}\|\,\|A\|}_{\kappa(A)}\;\frac{\|Ab\|\,\|x\|}{\|Ab\|\,\|x\|}$$

$$(y + \Delta y) = A(x + \Delta x)$$

$$y + \Delta y = Ax + A\Delta x$$

$$y = Ax$$

true is x

error is Δx

bound Δy with resp. y

worst case: Δx magnified most

$\Delta y = A \Delta x$ $\quad$ x magnified least

Demo: Condition number visualized

Demo: Conditioning of 2×2 Matrices