Matrix Conditioning

$$\kappa(A) = \|A\|_2 \underline{\|A^{-1}\|}_2$$

$$\text{conditioning} = \max_{\text{true}} \max_{\text{err. input}} \left(\frac{\text{rel err out}}{\text{rel err inp}} \right)$$

Conditioning

Now, let's study condition number of solving a linear system

$$Ax = b.$$

$$\underbrace{x = \hat{x} + \Delta x}_{\text{out.}} \qquad \underbrace{b = \hat{b} + \Delta b}_{\text{inp.}}$$

$$\max_{x,\Delta x} \left(\frac{\|\Delta x\| / \|x\|_2}{\|\Delta b\| / \|b\|_2} \right) \le \|A\|_2 \|A^{-1}\|_2$$

$$A\hat{x} = \hat{b}$$

distance $\hat{x}$ and x

Demo: Condition number visualized
Demo: Conditioning of 2×2 Matrices

More Properties of the Condition Number

What is $\operatorname{cond}(A^{-1})$?

$\kappa(A) = \|A\| \|A^{-1}\| = \|A^{-1}\| \|A\| = \kappa(A^{-1})$

What is the condition number of applying the matrix-vector multiplication $A\boldsymbol{x} = \boldsymbol{b}$? (I.e. now $\boldsymbol{x}$ is the input and $\boldsymbol{b}$ is the output)

Find b, $A^{-1}b = x$

condition number is $\kappa(A)$

Matrices with Great Conditioning (Part 1)

Give an example of a matrix that is *very* well-conditioned.
(I.e. has a condition-number that's *good* for computation.)
What is the best possible condition number of a matrix?

$\kappa(A \cdot A^{-1}) = \kappa(I) = 1$

$\hookrightarrow \|I\| \, \|I\| = 1$

$\|$

$\|AA^{-1}\| \cdot \|A^{-1}A\| \geq 1$

$1 = \|AA^{-1}\| \leq \|A\| \cdot \|A^{-1}\|$

$1 = \|A^{-1}A\| \leq \|A^{-1}\| \cdot \|A\|$

Matrices with Great Conditioning (Part 2)

What is the 2-norm condition number of an orthogonal matrix A?

$$Q^T Q = I \Rightarrow Q^{-1} = Q^T$$

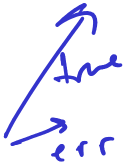
$$\|Q\|_2 = \max_{\|x\|_2=1} \sqrt{x^T \underbrace{Q^T Q}_{I} x} = \sqrt{x^T x}$$

$$= \|x\|_2 = 1$$

$$\kappa_2(Q) = \|Q\|_2 \, \|Q^T\|_2 = 1 \cdot 1 = 1$$

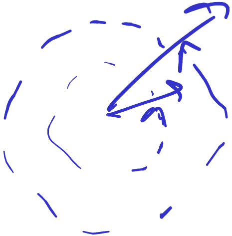
orthogonal

In-class activity: Matrix Conditioning



have = true + err

$$\max_{\text{true}} \max_{\text{err}} \frac{\text{amplification of err}}{\text{amplification of true}}$$



Outline

Solving Systems of Equations

Want methods/algorithms to solve linear systems. Starting small, a kind of system that's easy to solve has a ... matrix.

$$2x_1 + 3x_2 = 7$$

$$4x_2 = 1$$

$$x_2 = \frac{1}{4}$$

$$\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

Triangular Matrices

Solve

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

$$x_4 = b_4 / a_{44}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 - a_{14} x_4 \\ b_2 - a_{24} x_4 \\ b_3 - a_{34} x_4 \end{pmatrix}$$

Demo: Back-substitution

In-class activity: Forward-substitution

General Matrices

What about non-triangular matrices?

LU decomposition = Gaussian elimination

Given $n \times n$ matrix A, obtain lower triangular matrix L and upper triangular matrix U such that $A = LU$.

Is there some redundancy in this representation?

$$\mathrm{nnz}(L) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\mathrm{nnz}(L) + \mathrm{nnz}(U) = n(n+1)$$

L has unit diagonal $L_{ii} = 1$

Using LU Decomposition to Solve Linear Systems

Given $A = LU$, how do we solve $Ax = b$?

$L\underbrace{Ux}_{y} = b$

$Ly = b \Leftarrow$ solve for y fwd. subs.

$Ux = y \Leftarrow$ solve for x bwd subs.

2-by-2 LU Factorization (Gaussian Elimination)

Lets consider an example for $n = 2$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & \\ l_{21} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & u_{22} \end{pmatrix}$$

$a_{11} = 1 u_{11}$ so $u_{11} = a_{11}$

$a_{12} = 1 \cdot u_{12}$ so $u_{12} = a_{12}$

first row of U = first row of A

$l_{21} \cdot u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/u_{11}$

$$a_{22} = \begin{bmatrix} l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \end{bmatrix}$$

$$= l_{21} u_{12} + u_{22}$$

$$u_{22} = \underbrace{a_{22} - l_{21} u_{12}}_{\text{Schur complement}}$$

General LU Factorization (Gaussian Elimination)

$$A = \begin{bmatrix} a_{11} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & A_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \mathbf{l_{21}} & \underbrace{L_{22}}_{n-1} \end{bmatrix}^{n-1} \cdot \begin{bmatrix} u_{11} & \mathbf{u_{12}} \\ 0 & \underbrace{U_{22}}_{n-1} \end{bmatrix}^{n-1}$$

$$[u_{11} \quad u_{12}] = [a_{11} \quad a_{12}]$$

$$l_{21} \cdot u_{11} = a_{21}$$

$$l_{21} = a_{21} / u_{11}$$

$$\bar{A}_{22} = A_{22} - l_{21} \cdot u_{12}$$

$$[L_{22}, U_{22}] = LU\text{-alg}(\bar{A}_{22})$$