

LU Factorization

- solve linear systems

$$- A = LU \quad \square = \triangle \cdot \square$$

↑ ↑
lower tri upper tri

$$Ax = b \Rightarrow L(Ux) = b$$

General LU Factorization (Gaussian Elimination)

$$A = \left[\begin{array}{c|c} a_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & A_{22} \end{array} \right] = \left[\begin{array}{cc} 1 & 0 \\ \mathbf{l}_{21} & L_{22} \end{array} \right] \cdot \left[\begin{array}{cc} u_{11} & \mathbf{u}_{12} \\ 0 & U_{22} \end{array} \right]$$

1 n-1

$$[a_{11} \quad \mathbf{a}_{12} \quad -] = 1 \cdot [u_{11} \quad \mathbf{u}_{12} \quad -]$$

$$a_{21} = l_{21} \cdot u_{11} \Rightarrow l_{21} = a_{21} / u_{11}$$

$$| \quad = | \dots$$

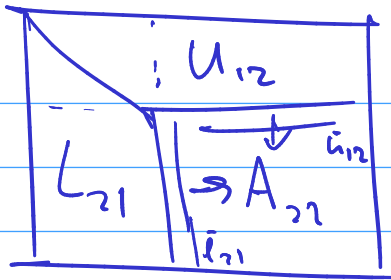
$$S_{22} = A_{22} - l_{21} \cdot \mathbf{u}_{12} \quad \square - |$$

$$[L_{22}, U_{22}] = LU(S_{22})$$

$$A = \begin{bmatrix} 1 & & \\ & L_{21} & L_{22} \\ & & 1 \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} & \\ & U_{22} & \\ & & \end{bmatrix}$$

$$\boxed{A_{22}} - \begin{array}{c} \longleftarrow U_{12} \\ | \\ L_{21} \end{array} \quad \Bigg| \quad \boxed{\phantom{A_{22}}} - \boxed{\phantom{A_{22}}}$$

$$\boxed{A_{22}} - \boxed{\phantom{A_{22}}} \boxed{\phantom{A_{22}}} \quad \Bigg| \quad \phantom{\boxed{\phantom{A_{22}}}}$$



$$S_{22} = A_{22} - L_{21} U_{12}$$

\bar{u}_{12} = first row of S_{22}

$$\bar{l}_{21} = S_{22} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} / S_{22} [1, 1]$$

Demo: Gaussian Elimination

LU: Failure Cases?

Is LU/Gaussian Elimination bulletproof?

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

What can be done to get something *like* an LU factorization?

$$\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} u_{11} \cdot 1 &= 0 \\ \underbrace{u_{11} \cdot l_{21}}_0 + \underbrace{1 \cdot 0}_0 &= 2 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

Partial Pivoting Example

Lets try to get an pivoted LU factorization of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = PA$$

\uparrow \swarrow

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$LU(PA) \rightarrow \begin{matrix} L \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix} \cdot \begin{matrix} U \\ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \end{matrix}$$

Permutation Matrices

How do we capture 'row swaps' in a factorization?

$$v := \begin{pmatrix} 0 \\ 5 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \\ 5 \\ 0 \\ 0 \end{pmatrix} := w$$

$$w = P v \quad \Bigg| \quad P = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 0 & 0 & 1 & \\ & & 0 & 1 & 0 & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 \end{bmatrix}$$

General LU Partial Pivoting

What does the overall process look like?

$$\begin{pmatrix} 2 & 4 & 7 \\ 1 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{P} \begin{pmatrix} 3 & 6 & 9 \\ 1 & 5 & 8 \\ 2 & 4 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ \frac{1}{3} & & \\ \frac{2}{3} & & \end{pmatrix} \quad U = \begin{pmatrix} 3 & 6 & 9 \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \begin{pmatrix} 6 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

$$PA = LU$$

$$\begin{pmatrix} 1 & \\ & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & u_{22} \end{pmatrix}$$

$$S_{22} = A_{22} - l_{21} u_{12}$$

$$P_S S_{22} = L_{22} U_{22}$$

$$\bar{A} = PA \quad \bar{A} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ a_{21} & P_S \bar{A}_{22} \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$$P_S S = A_{n \times n} L_{n \times n}$$

$$P_S := (n-1) \times (n-1)$$

$$P = P_1 \begin{bmatrix} 1 & & \\ & & \\ & & P_S \end{bmatrix}$$

$$PA = LU$$

P is invertible (full rank)

$$P = P^T ?$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

P is binary

$$P^T = P^{-1}$$

$$PA = LU \quad \Rightarrow \quad A = P^{-1}LU$$

$$Ax = b$$

$$A = P^{-1}LU$$

$$P^{-1}LUx = b$$

~~$$P^{-1}LUx = Pb$$~~

$$LUx = \underbrace{Pb}_{\bar{b}}$$


permute eqn $\bar{b} = Pb$

solve $Ly = Pb = \bar{b}$ \leftarrow fwd. subs.


solve $Ux = y$ \leftarrow back. subs.

Computational Cost


What is the computational cost of multiplying two $n \times n$ matrices?

 $C_{ij} = \sum_{k=1}^n A_{ik} B_{kj} \quad 2n^3 - n^2$

What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

 $n \times n \Rightarrow 2n^2$ operations

$T_{LU} \approx \sum_{i=1}^{n-1} 2i^2 \approx \frac{2}{3}n^3$

 $L \quad U$

mults + adds

More cost concerns

What's the cost of solving $Ax = b$?

$$O(n^3)$$

$$O(n^3) \text{ to get } PA = LU$$

$$O(n^2) \text{ to solve } Ax = b \quad LUx = Pb$$

What's the cost of solving $Ax_1 = b_1, \dots, Ax_n = b_n$?

$$Ax = B$$

$$O(n^3) \text{ for } A = LU \quad \left| \begin{array}{l} n \text{ - fwd/back subs} \\ n \text{ - permutation} \end{array} \right| \begin{array}{l} \text{overall} \\ O(n^3) \end{array}$$

What's the cost of finding A^{-1} ?

$$A \overset{\text{solve}}{\downarrow} X = I$$

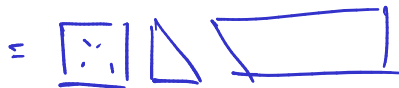
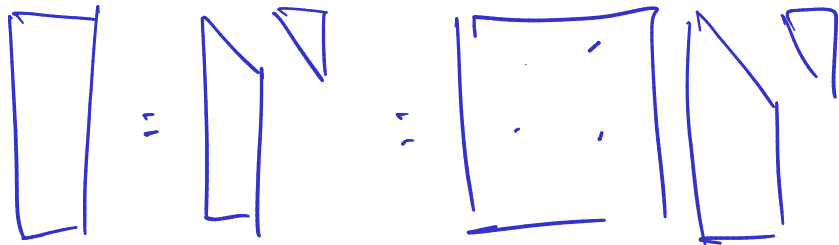
$$X = A^{-1}$$

$$\boxed{LU} \cdot \boxed{U^{-1}} = I$$

$$(LU)^{-1} = U^{-1}L^{-1}$$

LU: Rectangular Matrices

Can we compute LU of an $m \times n$ rectangular matrix?



Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
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The 'Undo' Button for Linear Operations: LU
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Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares
SVD: Applications
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 Data Fitting
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Iteration and Convergence
Solving One Equation
Solving Many Equations
Finding the Best: Optimization in 1D
Optimization in n Dimensions

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- ▶ $x \neq \mathbf{0}$ is called an **eigenvector** of A if there exists a λ so that

$$Ax = \lambda x.$$

- ▶ In that case, λ is called an **eigenvalue**.
- ▶ By this definition if x is an eigenvector then so is αx , therefore we will usually seek normalized eigenvectors, so $\|x\|_2 = 1$.

Finding Eigenvalues

How do you find eigenvalues?

$$Ax = \lambda x$$

$$(A - I\lambda)x = 0$$

$$\det(A - I\lambda)$$

instead approximate λ

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$= ad - bc$$

$$\det(AB) = \det(A) \det(B)$$

Distinguishing eigenvectors

Assume we have normalized eigenvectors x_1, \dots, x_n with eigenvalues $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Show that the eigenvectors are linearly independent.

$$0 = \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n$$

multiply by A many times

$$y^{(k)} = A^k \cdot (\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n)$$

$$\lim_{k \rightarrow \infty} y^{(k)} = \lim_{k \rightarrow \infty} \alpha_1 \lambda_1^k x_1 + \alpha_2 \lambda_2^k x_2 + \dots + \alpha_n \lambda_n^k x_n$$

$$y^{(k)} = y^{(k)} / \lambda_1^k$$

$$\lim_{k \rightarrow \infty} \|y^{(k)}\|_2 = \left\| \alpha_1 x_1 + \alpha_2 \frac{\lambda_2^k}{\lambda_1^k} x_2 + \dots + \alpha_n \frac{\lambda_n^k}{\lambda_1^k} x_n \right\|_2$$

↑
↑
lead to zero

$$= \|\alpha_1 x_1\| = \alpha_1 \|x_1\| = \alpha_1$$