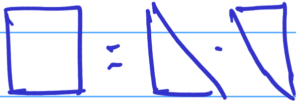
LU Factorization

- solve linear systems

- $A = \underbrace{L}_{\text{lower tri}}\underbrace{U}_{\text{upper tri}}$



$$Ax = b \Rightarrow L\overbrace{(Ux)}^{y} = b$$

General LU Factorization (Gaussian Elimination)

$$A = \left[\begin{array}{c|c} a_{11} & \mathbf{a_{12}} \\ \hline \mathbf{a_{21}} & A_{22} \end{array}\right] = \begin{bmatrix} 1 & 0 \\ \mathbf{l_{21}} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & \mathbf{u_{12}} \\ 0 & U_{22} \end{bmatrix}$$

$1 \quad n-1$

$$[a_{11} - a_{12} -] = 1 \cdot [u_{11} - u_{12} -]$$

$$a_{21} = l_{21} \cdot u_{11} \Longrightarrow l_{21} = a_{21} / u_{11}$$

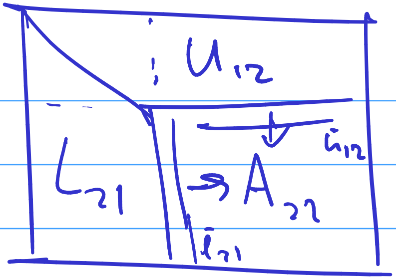
$$S_{22} = A_{22} - l_{21} \cdot u_{12}$$

$$[L_{22}, U_{22}] = LU(S_{22})$$

$$A = \begin{bmatrix} 1 & \\ l_{21} & L_{22} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ & U_{22} \end{bmatrix}$$

$$\boxed{A_{22}} - l_{21} u_{12}$$

$$\boxed{A_{22}} -$$



$S_{22} = A_{22} - L_{21} U_{12}$

$\bar{u}_{12}$ = first row of S_{22}

$\bar{l}_{21} = S_{22}[:, 1] / S_{22}[1, 1]$

Demo: Gaussian Elimination

LU: Failure Cases?

Is LU/Gaussian Elimination bulletproof?

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

What can be done to get something *like* an LU factorization?

$$\begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

$$u_{11} \cdot 1 = 0$$

$$\underbrace{u_{11} \cdot \ell_{21}}_{0} + \underbrace{1 \cdot 0}_{0} = 2$$

Partial Pivoting Example

Lets try to get an pivoted LU factorization of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = PA$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$LU(PA) \rightarrow \overset{L}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \cdot \overset{U}{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}$$

Permutation Matrices

How do we capture 'row swaps' in a factorization?

$$v = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ b \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \\ a \\ 0 \end{pmatrix} = w$$

$$w = Pv$$

$$P = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & 1 & \\ & & 0 & 1 & 0 & \\ & & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 \end{bmatrix}$$

General LU Partial Pivoting

What does the overall process look like?

$$\begin{pmatrix} 2 & 4 & 7 \\ 1 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow{P} \begin{pmatrix} 3 & 6 & 9 \\ 1 & 5 & 8 \\ 2 & 4 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & & \\ 1/3 & & \\ 2/3 & & \end{pmatrix} \qquad U = \begin{pmatrix} 3 & 6 & 9 \\ & & \\ & & \end{pmatrix}$$

$$\begin{pmatrix} 5 & 8 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \begin{pmatrix} 6 & 9 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix}$$

$$PA = LU \qquad \begin{pmatrix} 1 & \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ & U_{22} \end{pmatrix}$$

$$S_{22} = A_{22} - l_{21} u_{12}$$

$$P_S S_{22} = L_{22} U_{22}$$

$$\bar{A} = PA \qquad \bar{\bar{A}} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & P_S \bar{A}_{22} \end{pmatrix}$$

$$P_1 = \begin{array}{cc} & n \\ n & \begin{pmatrix} & & & 1 \\ & \ddots & & \\ & & \ddots & \\ 1 & & & \ddots \end{pmatrix} \end{array}$$

$P_S S = A_{22} L_{22}$ $\qquad$ P_S is $(n-1)$-by-$(n-1)$

$$P = P_1 \begin{bmatrix} 1 & \\ & P_S \end{bmatrix}$$

$PA = LU$

P is invertible (full rank)

$P = P^T$?

$$\begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & \end{bmatrix}$$

P is binary

$P^T = P^{-1}$

$$PA = LU \implies A = P^{-1} L U$$

$$Ax = b \qquad A = P^T L U$$

$$P^T L U x = b$$

$$\cancel{P P^T} L U x = P b$$

$$L U x = \underbrace{Pb}$$

permute to get $\bar{b} = P b$

solve $Ly = Pb = \bar{b}$ $\leftarrow$ fwd. subs.

solve $Ux = y$ $\leftarrow$ bwd. subs.

Computational Cost

What is the computational cost of multiplying two $n \times n$ matrices?

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad 2n^3 - n^2$$

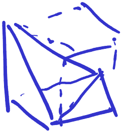
(i,j,k)

What is the computational cost of carrying out LU factorization on an $n \times n$ matrix?

$\Rightarrow 2n^2$ operations

$$T_{LU} \approx \sum_{i=1}^{n-1} 2i^2 \approx \frac{2}{3}n^3$$

mults + adds



More cost concerns

What's the cost of solving $A\boldsymbol{x} = \boldsymbol{b}$? $O(n^3)$

$O(n^3)$ to get $PA = LU$

$O(n^2)$ to solve $Ax = b$ $LUx = Pb$

What's the cost of solving $A\boldsymbol{x_1} = \boldsymbol{b_1}, \ldots, A\boldsymbol{x_n} = \boldsymbol{b_n}$? $AX = B$

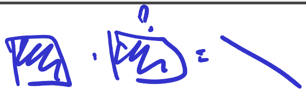
$O(n^3)$ for $A = LU$ | n - fwd/bwd subs | overall $O(n^3)$

n - permutation

What's the cost of finding A^{-1}?

solve

$AX = I$

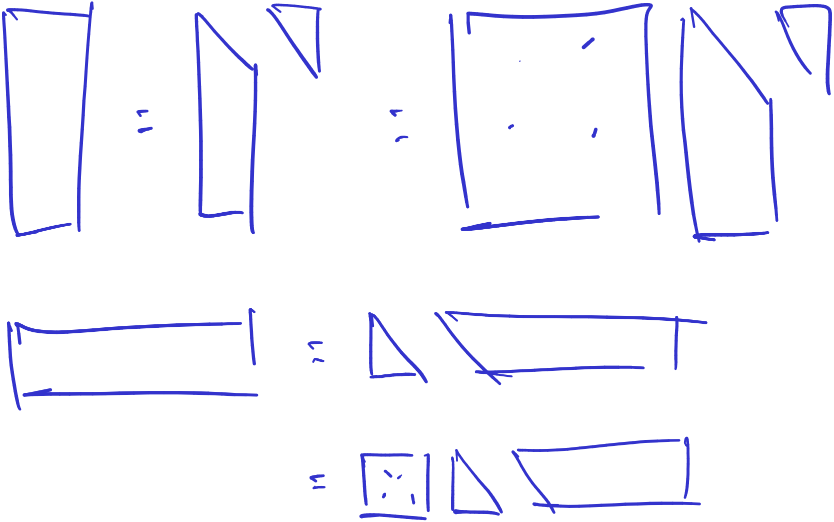


$X = A^{-1}$

$(LU)^{-1} = U^{-1} L^{-1}$

LU: Rectangular Matrices

Can we compute LU of an $m \times n$ rectangular matrix?



Outline

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \boldsymbol{0}$ is called an eigenvector of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}.$$

- In that case, λ is called an eigenvalue.
- By this definition if $\boldsymbol{x}$ is an eigenvector then so is $\alpha\boldsymbol{x}$, therefore we will usually seek normalized eigenvectors, so $\|\boldsymbol{x}\|_2 = 1$.

Finding Eigenvalues

How do you find eigenvalues?

$$Ax = \lambda x$$

$$(A - I\lambda)x = 0$$

$$\det(A - I\lambda)$$

$$\det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) =$$

$$= ad - bc$$

$$\det(AB) = \det(A) \cdot \det(B)$$

instead Approximate λ

Distinguishing eigenvectors

Assume we have normalized eigenvectors $x_1, \dots, x_n$ with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Show that the eigenvectors are linearly independent.

$$0 = \underbrace{\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n}_{\text{multiply by } A \text{ many times}}$$

$$y^{(k)} = A^k \cdot (\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n)$$

$$\lim_{k \to \infty} y^{(k)} \quad \lim_{k \to \infty} \alpha_1 \underbrace{\lambda_1^k}\, x_1 + \alpha_2 \lambda_2^k x_2 + \dots \alpha_n \lambda_n$$

$$\bar{y}^{(k)} = y^{(k)} / \lambda_1^k$$

$$\lim_{k\to\infty} \|\bar{y}^{(k)}\|_2 = \left\| \alpha_1 x_1 + \alpha_2 \frac{\lambda_2^k}{\lambda_1^k} x_2 + \dots \alpha_n \frac{\lambda_n^k}{\lambda_1^k} x_n \right\|_2$$

tend to zero

$$= \|\alpha_1 x_1\| = \alpha_1 \|x_1\| = \alpha_1$$