Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq \mathbf{0}$ is called an eigenvector of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda \boldsymbol{x}.$$

- In that case, λ is called an eigenvalue.
- By this definition if $\boldsymbol{x}$ is an eigenvector then so is $\alpha \boldsymbol{x}$, therefore we will usually seek normalized eigenvectors, so $\|\boldsymbol{x}\|_2 = 1$.

$x_1 \quad \lambda_1$

$x_2 \quad \lambda_2$

$x_3 \quad \lambda_3$

$\lambda_1 > \lambda_2 > \lambda_3$

$x_3 = \alpha_1 x_1 + \alpha_2 x_2$

$A^k x_3 = A^k(\alpha_1 x_1 + \alpha_2 x_2)$

$$\lambda_3^k x_3 = \lambda_1^k \alpha_1 x_1 + \lambda_2^k \alpha_2 x_2$$

$$\lim_{k \to \infty} \frac{A^k x_3}{\lambda_3^k} = \frac{\lambda_1^k}{\lambda_3^k} \alpha_1 x_1 + \frac{\lambda_2^k}{\lambda_3^k} \alpha_2 x_2$$

$$\begin{array}{c} \shortparallel \\ \left(\frac{\lambda_1}{\lambda_3}\right)^k \end{array}$$

$$\|x_3\| = \left(\frac{\lambda_1}{\lambda_3}\right)^k \alpha_1 \|x_1\| + \dots$$

eigenvectors with diff. eigvals
- linearly independent
- ~~orthogonal~~

$A = A^T$ symmetric

$x_1 \lambda_1 \quad x_2 \lambda_2$

$x_1^T A$
$= x_1^T A^T$
$= (A x_1)^T$
$= \lambda_1 x_1^T$

$$x_1^T (A x_2) = x_1^T \lambda_2 x_2 = \lambda_2 \underbrace{x_1^T x_2}$$

$$(x_1^T A) x_2 = \lambda_1 \underbrace{x_1^T x_2}$$

Distinguishing eigenvectors

Assume we have normalized eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ with eigenvalues $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$. Show that the eigenvectors are linearly-independent.

Diagonalizability

If we have n eigenvectors with different eigenvalues, the matrix is diagonalizable.

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & & | \end{bmatrix} \qquad AX = \begin{bmatrix} | & & | \\ \lambda_1 x_1 & \cdots & \lambda_n x_n \\ | & & | \end{bmatrix}$$

$$AX = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \cdot \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{D}$$

$$AX = XD \Rightarrow A = XDX^{-1}$$

similarity transformation

A is similar to B
if $\exists X$ s.t $A = XBX^{-1}$

symmetric case
X is full rank
X is orthogonal
$\hookrightarrow X^T = X^{-1}$

$A = XDX^{-1} = XDX^T$

Are all Matrices Diagonalizable?

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$$

$$x+y = \lambda x \qquad y=0$$

$$y = \lambda y \Rightarrow \lambda = 1$$

Power Iteration

We can use linear-independence to find the eigenvector with the largest eigenvalue. Consider the eigenvalues of A^{1000}.

$\lambda_1 > \dots \lambda_n$ of A

$x_1 \dots x_n$ eig. vecs

$A^{1000} x_1 = A^{999}(A^1 x_1) = \lambda_1 A^{999} x_1$

$= \lambda_1^{1000} x_1$

$\lambda_1^{1000} \dots \lambda_n^{1000}$ are eig. vals of A^{1000}

y - random vector

$$y = \alpha_1 x_1 + \dots + \alpha_n x_n$$

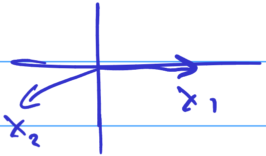
$$Ay = \alpha_1 \lambda_1 x_1 + \dots + \alpha_n \lambda_n x_n$$

$$\frac{A^{1000} y}{\lambda_1^{1000}} = \frac{\alpha_1 \lambda_1^{1000} x_1 + \dots + \alpha_n \lambda_n^{1000} x_n}{\lambda_1^{1000}}$$

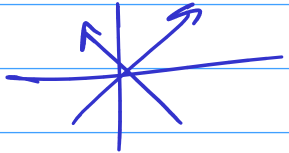
arbitrary vector

normalized eig-vectors

$X := 3(x_1 + x_2)$



symmetric



Power Iteration: Issues?

What could go wrong with Power Iteration?

overflow

↳ fixed by normalization

normalized Power iteration

if $\lambda_1 = \lambda_2$ e.g. vecs x_1, x_2

$\bar{x} = \alpha_1 x_1 + \alpha_2 x_2$ | $A\bar{x} = \lambda_1 \bar{x}$

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$\frac{x^T A x}{x^T x}$$

Rayleigh quotient

Convergence of Power Iteration

What can you say about the convergence of the power method? Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$

$$v_1^{(k+1)} = \frac{A v_1^{(k)}}{\|v_1^{(k)}\|} = \frac{A(\alpha_1 x_1 + \alpha_2 x_2 + \dots)}{\|v_1^{(k)}\|}$$

$$= \frac{\lambda_1 \left(\alpha_1 x_1 + \alpha_2 \frac{\lambda_2}{\lambda_1} x_2 + \dots\right)}{\|v_1^{(k)}\|}$$

convergence $e_k = O\left(\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)$ associated with $\frac{\lambda_2}{\lambda_1}$

Transforming Eigenvalue Problems

Suppose we know that $A\boldsymbol{x} = \lambda\boldsymbol{x}$. What are the eigenvalues of these changed matrices?

Power. $A \to A^k$

λ^k

Shift. $A \to A - \sigma I$

$(A - \sigma I)x = Ax - \sigma I x = \lambda x - \sigma x = (\lambda - \sigma)x$

Inversion. $A \to A^{-1}$

$Ax = \lambda x$

$\lambda^{-1} x = A^{-1} x$

$A^{-1} A x = A^{-1} \lambda x \Rightarrow x = \lambda A^{-1} \lambda$

Inverse Iteration / Rayleigh Quotient Iteration

Describe inverse iteration. $\rightarrow$ finds eig-vec with smallest eig-val

$$x^{(k+1)} = A^{-1} x^{(k)}$$

solve $A x^{(k+1)} = x^{(k)}$

Describe Rayleigh Quotient Iteration. closest to σ

$$x^{(k+1)} = (A - \sigma I)^{-1} x^{(k)}$$

solve for $x^{(k+1)}$ in

$$(A - \sigma I) x^{(k+1)} = x^{(k)}$$