

Outline

Brief Example Overview

SVD

Review Definition

SVD Applications

linear systems

least squares — data fitting

approximation

Computing $\kappa(A)$, given AX , $A^{-1}X$
Estimating

$$\kappa(A) = \|A\|_2 \|A^{-1}\|_2$$

$$= \max_x \left(\frac{\|Ax\|_2}{\|x\|_2} \right)$$

maximum amplification
of a column in AX

$$\max_x \left(\frac{\|A^{-1}x\|_2}{\|x\|_2} \right)$$

—

Singular Value Decomposition

What is the Singular Value Decomposition ('SVD')?

$$A = U \Sigma V^T$$

← right singular vectors

↑ singular values

↑ left singular vectors

singular vectors

$$(A^T A) V = V \Sigma^2$$

$$A^T = V \Sigma U^T$$

U are the eigenvectors of AA^T

$$AA^T U = U \Sigma^2$$

Computing the SVD

How can I compute an SVD of a matrix A ?

Find the columns of V

↳ as the eigenvectors of $A^T A$

Diagonalize $A^T A$

$$A^T A V = V \Sigma^2$$

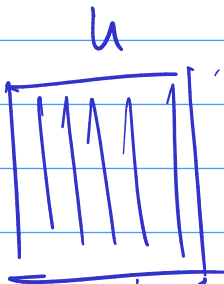
$$A = U \Sigma V^T$$

$$A V \Sigma^{-1} = U$$

↳ invertible?

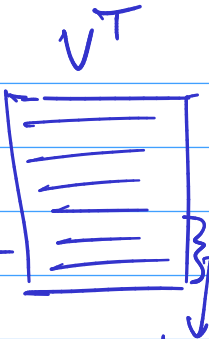
if σ_i (diagonal entry of Σ) is zero and A is square, then A is singular

A

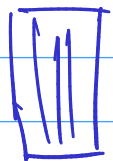


mult by row

ϵ



mult by row



Demo: Computing the SVD $\leftarrow n \times n$

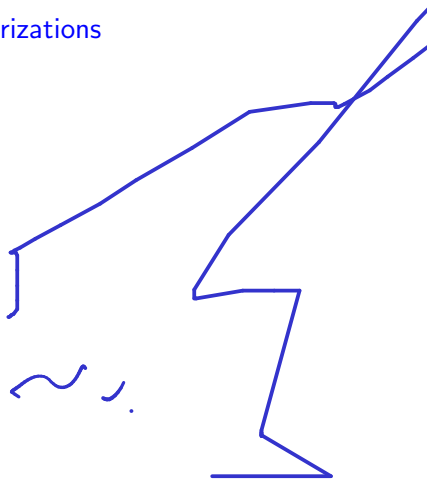
Diagonalization has cost $O(n^3)$

$A^T A V$ has cost $O(n^3)$

↑
constant
is relatively high

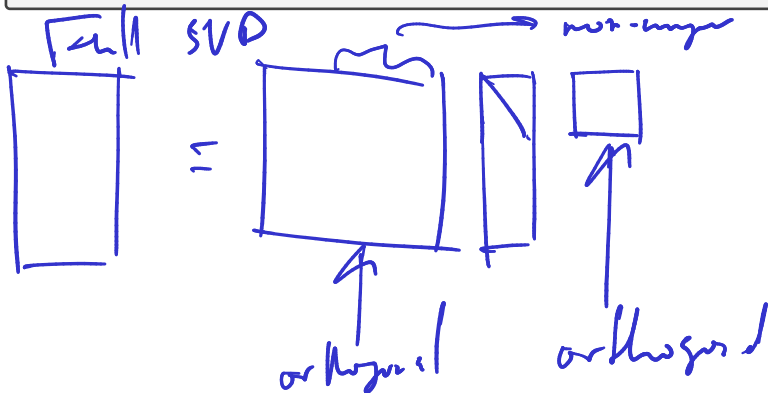
How Expensive is it to Compute the SVD?

Demo: Relative Cost of Matrix Factorizations

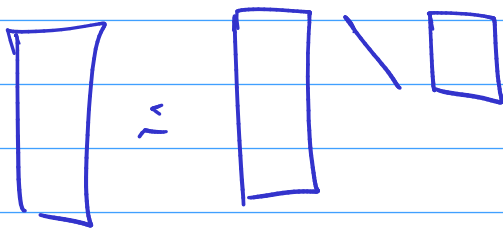


'Reduced' SVD

Is there a 'reduced' factorization for non-square matrices?



Reduced SVD



Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems
Data Fitting
Norms and Condition Numbers
Low-Rank Approximation

Interpolation

Iteration and Convergence

Solving One Equation

Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems
Data Fitting
Norms and Condition Numbers
Low-Rank Approximation

Interpolation

Iteration and Convergence

Solving One Equation

Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

Solve Square Linear Systems

Can the SVD $A = U\Sigma V^T$ be used to solve *square* linear systems?
At what cost (once the SVD is known)?

↪ is square

$$A = U\Sigma V^T$$

$$Ax = b \Rightarrow U\Sigma V^T x = b$$

↑
find

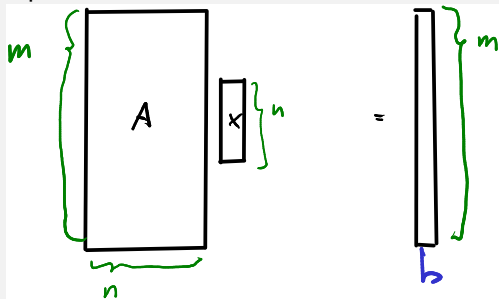
$$\Sigma V^T x = U^T b$$

$$V^T x = \Sigma^{-1} U^T b$$

$$x = \underbrace{V \Sigma^{-1} U^T}_{\sim A^{-1}} b$$

Tall and Skinny Systems

Consider a 'tall and skinny' linear system, i.e. one that has more equations than unknowns:



In the figure: $m > n$. How could we solve that?

Least square problem

Minimize residual, find x such that
 $Ax = b$ minimizes $\sum_{i=1}^m (b_i - b_i^e)^2$

Solving Least-Squares

How can I actually solve a least-squares problem $Ax \cong b$?

minimize $\|Ax - b\|_2^2 \sim$ minimize $\|Ax - b\|_2$

$$A = U \Sigma V^T$$

$$\|U \Sigma V^T x - b\|_2 \sim \|\underbrace{\Sigma V^T x}_{\text{minimize}} - U^T b\|_2$$

first minimize y

$$\begin{aligned} U \Sigma V^T x &\cong b \\ x &\cong V \Sigma^+ U^T b \end{aligned}$$

pseudoinverse

$$\Sigma_{ii}^+ = \begin{cases} 1/\Sigma_{ii} & \text{if } \Sigma_{ii} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$A = U \Sigma V^T$$

$$A^+ = V \Sigma^+ U^T$$

Square case

$$\begin{aligned} A^{-1} &= (U \Sigma V^T)^{-1} = V^{-T} \Sigma^{-1} U^{-1} \\ &= V \Sigma^{-1} U^T \\ &\approx V \Sigma^+ U^T \end{aligned}$$

In-class activity: SVD and Least Squares

The Pseudoinverse: A Shortcut for Least Squares

How could the solution process for $A\mathbf{x} \cong \mathbf{b}$ be with an SVDA = $U\Sigma V^T$ be 'packaged up'?

The Normal Equations

You may have learned the 'normal equations' $A^T A x = A^T b$ to solve $Ax \cong b$.

Why not use those?

$$\text{cond}(A^+ A) \sim \text{cond}(A)^2$$

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems
Data Fitting
Norms and Condition Numbers
Low-Rank Approximation

Interpolation

Iteration and Convergence

Solving One Equation

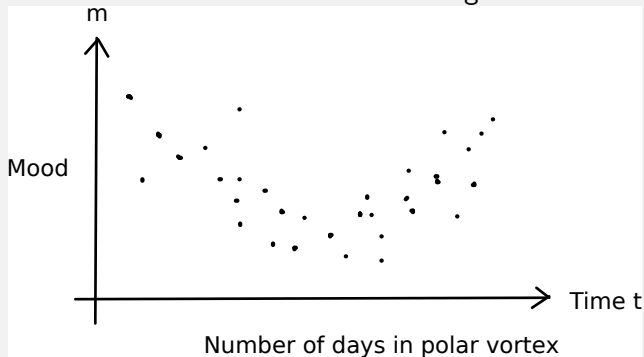
Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

Fitting a Model to Data

How can I fit a model to measurements? E.g.:



Demo: Data Fitting using Least Squares

Outline

Python, Numpy, and Matplotlib
Making Models with Polynomials
Making Models with Monte Carlo
Error, Accuracy and Convergence
Floating Point
Modeling the World with Arrays
 The World in a Vector
 What can Matrices Do?
 Graphs
 Sparsity
Norms and Errors
The 'Undo' Button for Linear Operations: LU
Repeating Linear Operations:
Eigenvalues and Steady States
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares

SVD: Applications

Solving Funny-Shaped Linear Systems
Data Fitting
Norms and Condition Numbers
Low-Rank Approximation

Interpolation

Iteration and Convergence

Solving One Equation

Solving Many Equations

Finding the Best: Optimization in 1D

Optimization in n Dimensions

Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

$$\sigma_{\max} = \|A\|_2 \quad \sigma_{\min} = \frac{1}{\max_i \sum_{j=1}^n \dots}$$

$$\|A\|_2 = \max_{\|x\|_2=1} \|A x\|_2$$

$$\text{cond}(A) = \frac{\sigma_{\max}}{\sigma_{\min}}$$

$$= \max_{\|y\|_2=1} \|U \Sigma y\|_2$$

$$= \max_{\|y\|_2=1} \|\Sigma y\|_2 = \max_i \sigma_{ii} = \sigma_{\max}$$