Outline

Brief Example Overview

SVD

  Review Definition

SVD Applications

  linear systems

  least squares — data fitting

  approximation

Computing (Estimating) $\kappa(A)$, given AX, $A^{-1}X$

$$\kappa(A) = \|A\|_2 \, \|A^{-1}\|_2$$

$$= \underbrace{\max_x \left(\frac{\|Ax\|_2}{\|x\|_2} \right)}_{\text{maximum amplification of a column in } AX}$$

$$\underbrace{\max_x \left(\frac{\|A^{-1}x\|_2}{\|x\|_2} \right)}_{\ldots}$$

Singular Value Decomposition

What is the Singular Value Decomposition ('SVD')?

$$A = U \Sigma V^T$$

V^T ← right singular vectors

Σ ↑ singular values

U ↑ left singular vectors

singular vectors

$$(A^T A) V = V \Sigma^2$$

$A^T = V \Sigma U^T$

U are the eigenvectors of AA^T

$$AA^T U = U \Sigma^2$$

Computing the SVD

How can I compute an SVD of a matrix A?

Find the columns of V

↳ as the eigenvectors of A^TA

Diagonalize A^TA

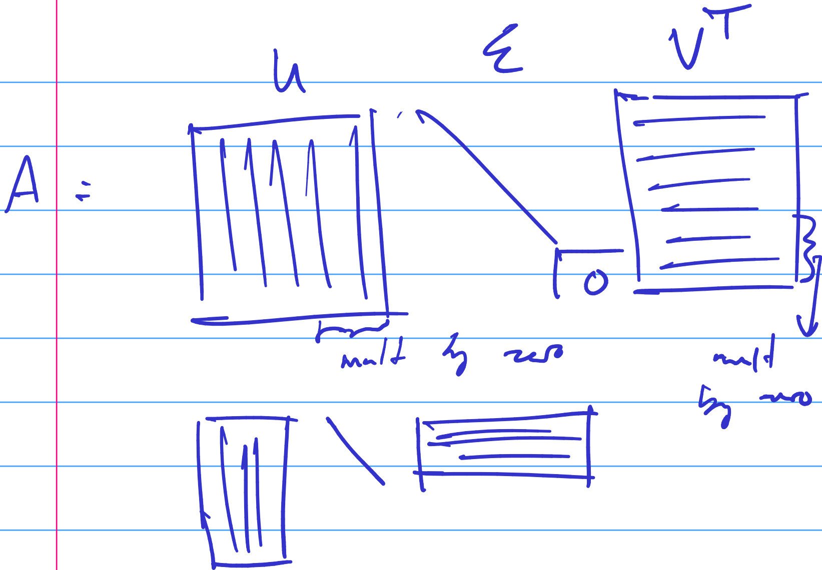
$$A^TAV = V\Sigma^2$$

if σ_i (diagonal entry of Σ) is zero and A is square, then A is singular

$$A = U\Sigma V^T$$

$$AV\Sigma^{-1} = U$$

invertible?


A =
U
Σ
V^T
0
mult by zero
mult by zero

Demo: Computing the SVD ← $n \times n$

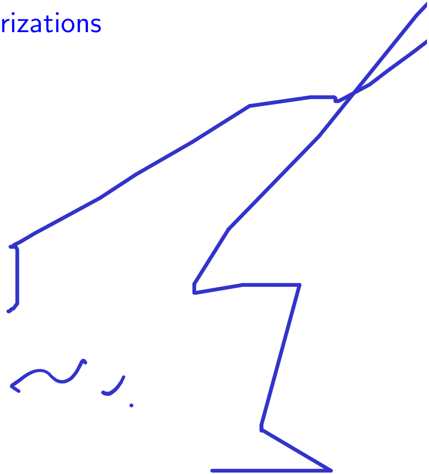
Diagonalization has cost $O(n^3)$

A^TAV has cost $O(n^3)$

↑ constant is relatively high

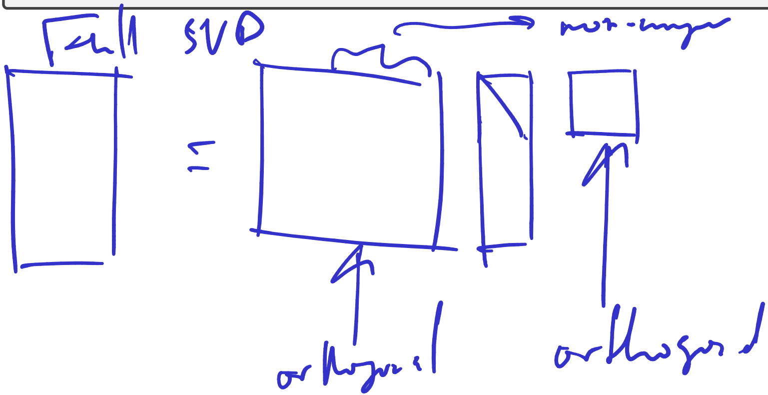
How Expensive is it to Compute the SVD?

Demo: Relative Cost of Matrix Factorizations

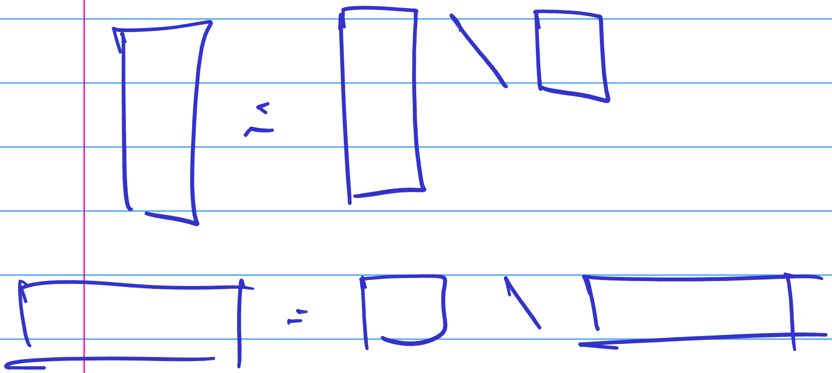


‘Reduced’ SVD

Is there a ‘reduced’ factorization for non-square matrices?



Reduced SVD



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Solve Square Linear Systems

Can the SVD $A = U\Sigma V^T$ be used to solve *square* linear systems?
At what cost (once the SVD is known)?

Σ is square

$$A = U\Sigma V^T$$

$$Ax = b \Rightarrow U\Sigma V^T x = b$$

find x

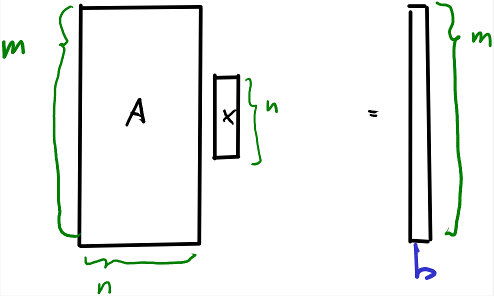
$$\Sigma V^T x = U^T b$$

$$V^T x = \Sigma^{-1} U^T b$$

$$x = \underbrace{V\Sigma^{-1}U^T}_{\sim A^{-1}} b$$

Tall and Skinny Systems

Consider a 'tall and skinny' linear system, i.e. one that has more equations than unknowns:



In the figure: $m > n$. How could we solve that?

Least square problem

Minimize residual, find x such that $A x = \bar{b}$ minimizes $\sum_{i=1}^{m} (\bar{b}_i - b_i)^2$

Solving Least-Squares

How can I actually *solve* a least-squares problem $Ax \cong \boldsymbol{b}$?

minimize $\|Ax - b\|_2^2$ ~ minimize $\|Ax - b\|_2$

$A = U\Sigma V^T$

minimize

$\|U\Sigma V^T x - b\|_2 \sim \|\Sigma V^T x - U^T b\|_2$

first minimizing y

$U\Sigma V^T x \cong b$

$x \cong V\Sigma^+ U^T b$

pseudoinverse

$\Sigma^+_{ii} = 1/\Sigma_{ii}$ if $\Sigma_{ii} \neq 0$

$= 0$ otherwise

$$A = U\Sigma V^T$$

$$A^+ = V\Sigma^+ U^T$$

Square case

$$A^{-1} = (U\Sigma V^T)^{-1} = V^{-T}\Sigma^{-1}U^{-1}$$

$$= V\Sigma^{-1}U^T$$

$$\approx V\Sigma^+U^T$$

In-class activity: SVD and Least Squares

The Pseudoinverse: A Shortcut for Least Squares

How could the solution process for $A\boldsymbol{x} \cong \boldsymbol{b}$ be with an SVD$A = U\Sigma V^T$ be 'packaged up'?

The Normal Equations

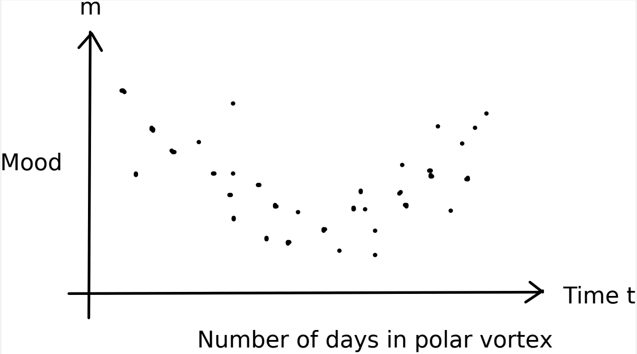
You may have learned the 'normal equations' $A^TAx = A^Tb$ to solve $Ax \cong b$.
Why not use those?

cond$(A^TA) \sim$ cond$(A)^2$

Outline

Fitting a Model to Data

How can I fit a model to measurements? E.g.:



Demo: Data Fitting using Least Squares

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Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

$\sigma_{max} = \|A\|_2$

$\sigma_{min} = 1 / \max_i \Sigma^+_{ii}$

$\text{cond}(A) = \frac{\sigma_{max}}{\sigma_{min}}$

$$\|A\|_2 = \max_{\|x\|=1} \|U \Sigma V^T x\|_2$$

$$= \max_{\|y\|=1} \|U \Sigma y\|_2$$

$$= \max_{\|y\|=1} \|\Sigma y\|_2 = \max_i \Sigma_{ii} = \sigma_{max}$$