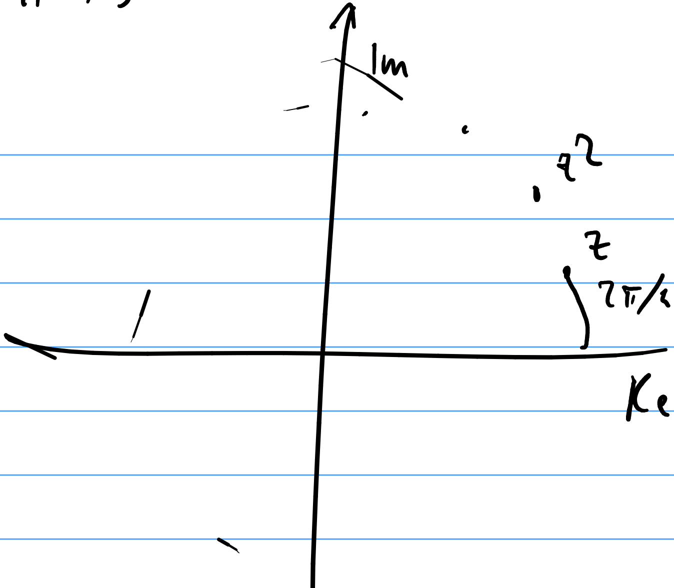
$e^{i\varphi} = \cos(\varphi) + i \sin \varphi$



$z = e^{2\pi i/n}$ $\quad n = 7 \quad |z| = 1$

Overview

- SVD
  - ↳ lsq
  - ↳ norms
  - ↳ PCA
- Interpolation

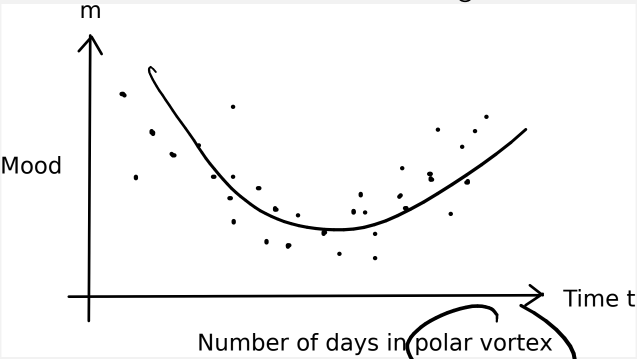
$$PA = LU$$

$$A = P^T L U$$

"PLU" factorization

Fitting a Model to Data

How can I fit a model to measurements? E.g.:



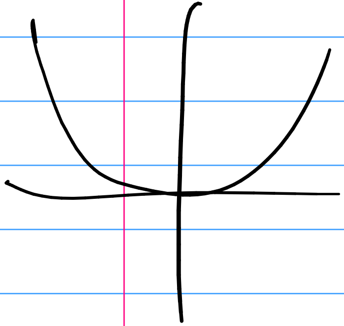
$$p(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2$$

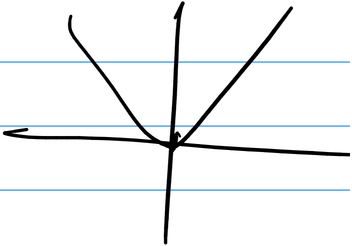
$$V \vec{\alpha} \approx \vec{y}$$

$$Ax \cong b \rightarrow \min_x \|Ax - b\|_2$$

l_2-minimization

l_1-





Demo: Data Fitting using Least Squares

Meaning of the Singular Values

What do the singular values mean? (in particular the first/largest one)

$$A = U \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} V^T$$

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2$$

$$= \max_{\|x\|_2 = 1} \| U \Sigma V^T x \|_2$$

$$= \max_{\|x\|_2 = 1} \|\Sigma V^T x\|_2$$

$\rightarrow \|x_2\| = 1$

$\|V^T x\|_2 = 1$

$$= \max_{\|V^T x\|_2 = 1} \|\Sigma V^T x\|_2$$

$$= \max_{\|y\|_2 = 1} \|\Sigma y\|_2$$

$\uparrow = V^T x$

$$= \|\Sigma\|_2 = \sigma_1$$

$$\| Qx \|_2^2 = \| x \|_2^2$$

Condition Numbers

How would you compute a 2-norm condition number?

$$\mathrm{cond}_2(A) = \|A\|_2 \cdot \|A^{-1}\|_2$$

$$= \sigma_1 \cdot \frac{1}{\sigma_n}$$

$$A^{-1} = V\Sigma^{-1}U^T = V\begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix}U^T$$

Outline

SVD as Sum of Outer Products

> What's another way of writing the SVD?

$m \times n$

$$A = U \Sigma V^T = \underbrace{\begin{pmatrix} | & & | \\ u_1 & \cdots & u_m \\ | & & | \end{pmatrix}}_{m \times m} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ 0 & & 0 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

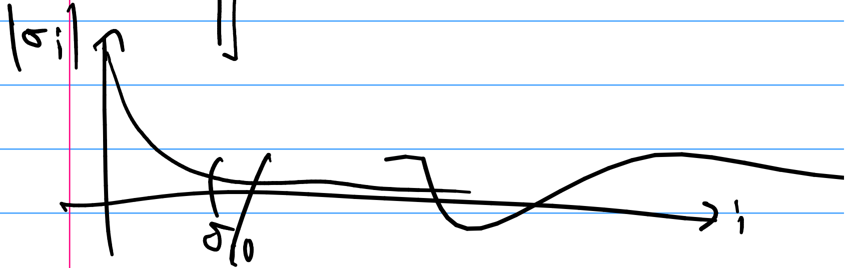
$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} - & \sigma_1 v_1 & - \\ & \vdots & \\ - & \sigma_n v_n & - \end{pmatrix}$$

$$= \sigma_1 u_1 v_1^T + \underbrace{\sigma_2 u_2 v_2^T}_{\text{rank } 1} + \cdots + \underbrace{\sigma_n u_n v_n^T}_{\text{rank } 1}$$

1

$$A_1 = \sigma_1 u_1 v_1^T$$

$$= \begin{bmatrix} u \end{bmatrix} \sigma_1 \begin{bmatrix} v_1 \end{bmatrix}$$



Low-Rank Approximation (I)

What is the *rank* of $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T$?

1

What is the *rank* of $\sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T$?

2

Demo: Image Compression

Low-Rank Approximation

What can we say about the low-rank approximation

$$A_k = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^T$$

to

$$A = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + \sigma_1 \boldsymbol{u}_n \boldsymbol{v}_n^T ?$$

For simplicity, assume $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$.

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F$$

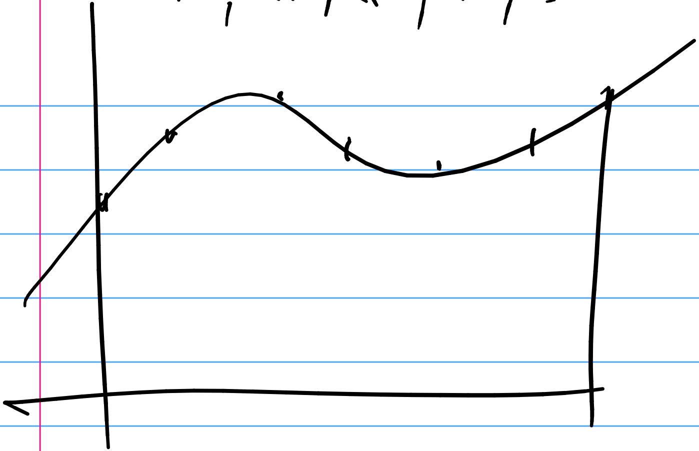
Eckart-Young Mirsky theorem

$$A = \begin{pmatrix} | & & | \\ \vec{a}_1 & \cdots & a_n \\ | & & | \end{pmatrix}$$

$$\approx \begin{pmatrix} u_1 & \cdots & u_{10} \end{pmatrix} \underbrace{\begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{10} \end{pmatrix} \begin{pmatrix} V^T \end{pmatrix}}_{}$$

Outline

$1, x, x^2, x^3, \ldots$



Recap: Interpolation

Starting point: Looking for a linear combination of functions φ_i to hit given data points (x_i, y_i).

Interpolation becomes solving the linear system:

$$y_i = f(x_i) = \sum_{j=0}^{N_{\text{func}}} \alpha_j \underbrace{\varphi_j(x_i)}_{V_{ij}} \qquad \leftrightarrow \qquad V\boldsymbol{\alpha} = \boldsymbol{y}.$$

Want unique answer: Pick $N_{\text{func}} = N \to V$ square.

V is called the (generalized) Vandermonde matrix.

Main lesson:

$$V \,(\text{coefficients}) = (\text{values at nodes})\,.$$