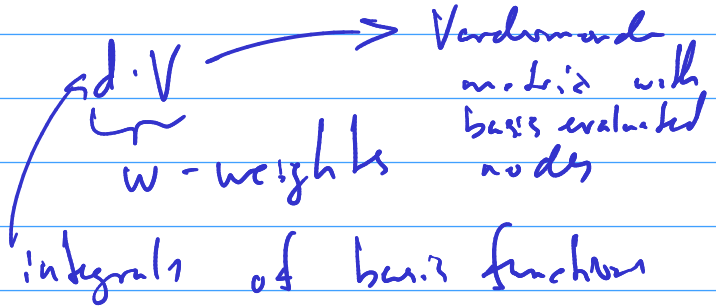


# Quadrature weights



# Outline

Python, Numpy, and Matplotlib  
Making Models with Polynomials  
Making Models with Monte Carlo  
Error, Accuracy and Convergence  
Floating Point  
Modeling the World with Arrays  
    The World in a Vector  
    What can Matrices Do?  
    Graphs  
    Sparsity  
Norms and Errors  
The 'Undo' Button for Linear Operations: LU  
Repeating Linear Operations:  
Eigenvalues and Steady States  
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares  
SVD: Applications  
    Solving Funny-Shaped Linear Systems  
    Data Fitting  
    Norms and Condition Numbers  
    Low-Rank Approximation  
Interpolation  
Iteration and Convergence  
**Solving One Equation**  
Solving Many Equations  
Finding the Best: Optimization in 1D  
Optimization in  $n$  Dimensions

# Solving Nonlinear Equations

What is the goal here?

given  $f$

$$\text{find } \underset{\uparrow}{f}(x) = 0$$

$$f(x) = g$$

$$\hat{f}(x) = f(x) - g$$

$$\text{Solve } \underset{\uparrow}{\hat{f}}(x) = 0$$

# Bisection Method



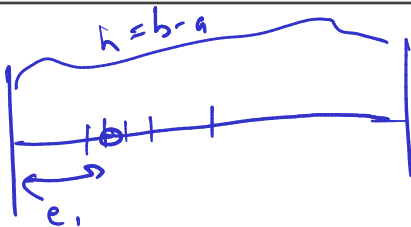
Assume continuous function  $f$  has a zero on the interval  $[a, b]$  and

$$\text{sign}(f(a)) = -\text{sign}(f(b)).$$

Perform binary search: check sign of  $f((a+b)/2)$  and define new search interval so that ends have opposite sign.

**Demo:** Bisection Method

What's the rate of convergence? What's the constant?

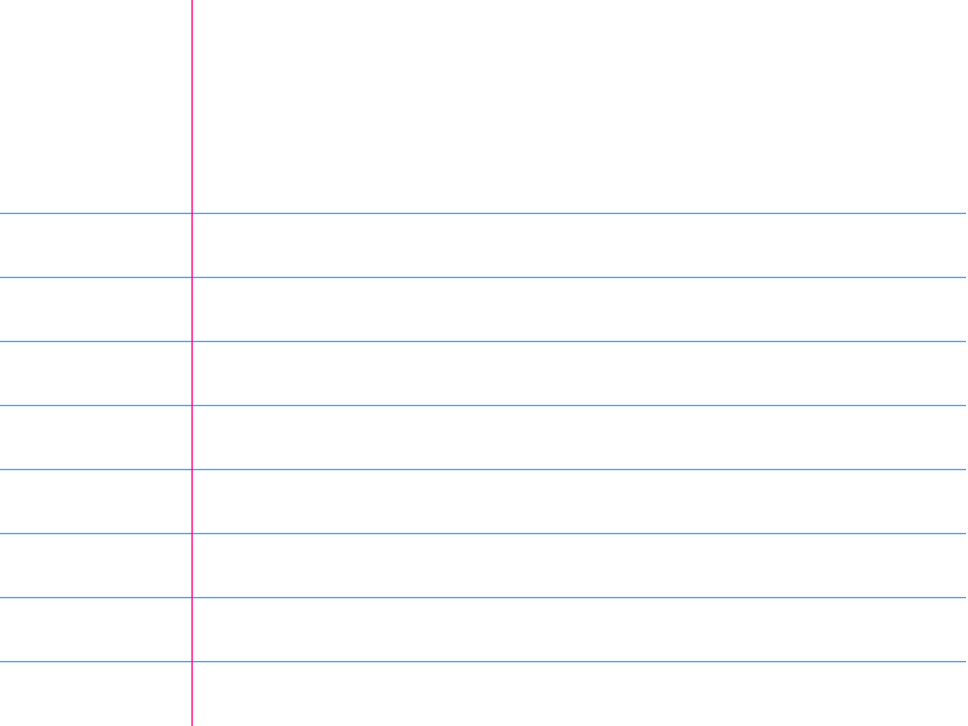


$$e_1 \leq h$$

$$e_1 \leq h/2$$

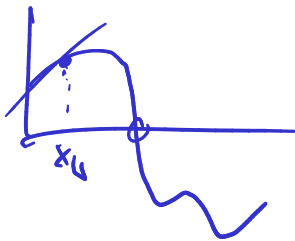
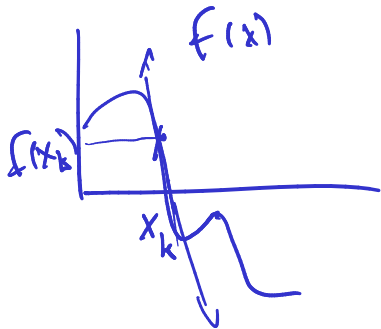
$$e_2 \leq h/4$$

linear



# Newton's Method

Derive Newton's method.



$$\tilde{f}_k(x) = 0$$

$$\Rightarrow 0 = f(x_k) + f'(x_k)(x - x_k)$$

$$f(x) \approx \underbrace{f(x_k) + f'(x_k)(x - x_k)}_{\tilde{f}_k(x)}$$

$$0 = f(x_k) + f'(x_k) x - f'(x_k) x_k$$

$$x = \frac{-f(x_k)}{f'(x_k)} + x_k$$

$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

$\uparrow$   
 $x_{k+1}$

**Demo:** Newton's method

**Demo:** Convergence of Newton's Method

What are some **drawbacks** of Newton?

may not converge  
stuck at local minima  
or move far away from last  
guess  
need derivative

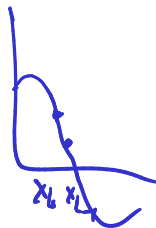
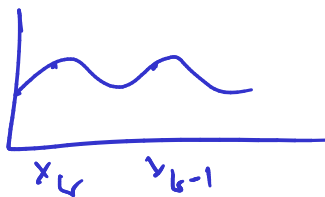


# Secant Method

What would Newton without the use of the derivative look like?

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

wand  $x_k \approx x_{k-1}$



$e_k = \text{error at iter } k$

Second  
→

$$\lim_{k \rightarrow \infty} \frac{e_{k+1}}{e_k e_{k-1}} = C$$

Side  
Note

$$\lim_{k \rightarrow \infty} \frac{e_{k+1}}{e_k^2} = C$$

## Secant Method Drawbacks

What are some **drawbacks** of Secant?

- 2 starting guesses
- convergence problem Newton

**Demo:** Secant Method

**In-class activity:** Secant Method

+ no derivative

# Outline

Python, Numpy, and Matplotlib  
Making Models with Polynomials  
Making Models with Monte Carlo  
Error, Accuracy and Convergence  
Floating Point  
Modeling the World with Arrays  
    The World in a Vector  
    What can Matrices Do?  
    Graphs  
    Sparsity  
Norms and Errors  
The 'Undo' Button for Linear Operations: LU  
Repeating Linear Operations:  
Eigenvalues and Steady States  
Eigenvalues: Applications

Approximate Undo: SVD and Least Squares  
SVD: Applications  
    Solving Funny-Shaped Linear Systems  
    Data Fitting  
    Norms and Condition Numbers  
    Low-Rank Approximation  
Interpolation  
Iteration and Convergence  
Solving One Equation  
**Solving Many Equations**  
Finding the Best: Optimization in 1D  
Optimization in  $n$  Dimensions

# Solving Nonlinear Equations

What is the goal here?

$$f(\vec{x}) = \vec{0}$$

---

$$f(\vec{x}) = \vec{b}$$

$$\tilde{f}(\vec{x}) = f(\vec{x}) - \vec{b}$$

$$f(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

# Newton's method

What does Newton's method look like in  $n$  dimensions?

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \dots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \dots & \frac{\partial f_n(x)}{\partial x_n} \end{bmatrix}$$



$$x_{k+1} = x_k - J_f(x_k)^{-1} f(x_k)$$