Outline

Boundary Value Problems

- Side conditions prescribing solution or derivative values at specified points are required to make solution of ODE unique
- For initial value problem, all side conditions are specified at single point, say t_0
- For *boundary value problem* (BVP), side conditions are specified at more than one point
- kth order ODE, or equivalent first-order system, requires k side conditions
- For ODEs, side conditions are typically specified at endpoints of interval $[a, b]$, so we have *two-point boundary value problem* with boundary conditions (BC) at a and b.

ODEs: Boundary Value Problems in 1D

Consider linear ODE of the form $\mathcal{L}\tilde{u} = f(x)$, with $\tilde{u}(x)$ satisfying given BCs.

Here, we consider three basic approaches to find $u \approx \tilde{u}$.

- **Finite difference methods (FDM):**
  - Essentially, approximate differential equation at multiple points, x_i, $i = 0, \ldots, n$.
  - **Note:** *we will use either n or $n+1$ points according to what makes the most sense in the given context.*

- **Collocation methods:**
  - Approximate the solution by an expansion,

$$u(x) \ := \ \sum_{j=0}^{n} u_j \phi_j(x),$$

  - Solve for coefficients u_j such that the ODE is satisfied at a chosen set of collocation points, x_i, along with the boundary conditions.
  - That is, the residual, $r(x) := (L\tilde{u} - Lu)$ is forced to be zero at x_i, $i = 0, \ldots, n$.

- **Weighted residual technique (WRT):**
  - Approximate the solution by an expansion,

$$u(x) \quad := \quad \sum_{j=0}^{n} u_j \phi_j(x),$$

  and solve for coefficients u_j such that the ODE is satisfied in some weighted sense.
  - That is, rather than enforcing $r(x) = 0$ at isolated points, we require $r(x)$ to be orthogonal to a set of weight functions, $\psi_i(x)$:

$$\int_a^b \psi_i(x)\, r(x)\, dx \quad = \quad \int_a^b \psi_i(x)\, L(u) - L(\tilde{u})\, dx \quad = \quad 0, \quad \text{or}$$

$$\int_a^b \psi_i(x)\, L(u) \quad = \quad \int_a^b \psi_i(x)\, L(\tilde{u})\, dx$$

  for $i = 0,\ 1,\ ...$
  - Note that if $\psi_i(x) = \delta(x - x_i)$ (Dirac delta function), we recover collocation.
  - Most often, the *test-space* and *trial space* are the same: $\psi_i := \phi_i$. (Galerkin case.)
  - Finite element, spectral, spectral element methods are examples of WRTs.

  - WRTs have many advantages over collocation in terms of flexibility of basis functions, application of boundary conditions, etc., and are generally preferred over collocation.

Finite Difference Method

- *Finite difference method* converts BVP into system of algebraic equations by replacing all derivatives with finite difference approximations
- For example, to solve two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

  with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

  we introduce mesh points $t_i = a + ih$, $i = 0, 1, \ldots, n+1$, where $h = (b-a)/(n+1)$
- We already have $y_0 = u(a) = \alpha$ and $y_{n+1} = u(b) = \beta$ from BC, and we seek approximate solution value $y_i \approx u(t_i)$ at each interior mesh point t_i, $i = 1, \ldots, n$

Finite Difference Method, continued

- We replace derivatives by finite difference approximations such as

$$u'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

- This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

  to be solved for unknowns y_i, $i = 1, \ldots, n$

- System of equations may be linear or nonlinear, depending on whether f is linear or nonlinear

Finite Difference Method, continued

- We replace derivatives by finite difference approximations such as

$$u'(t_i) \approx \frac{y_{i+1} - y_{i-1}}{2h} \longleftarrow \textit{Error is } O(h^2)$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \longleftarrow \textit{Error is } O(h^2)$$

- This yields system of equations

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = f\left(t_i, y_i, \frac{y_{i+1} - y_{i-1}}{2h}\right)$$

to be solved for unknowns y_i, $i = 1, \ldots, n$

- System of equations may be linear or nonlinear, depending on whether f is linear or nonlinear

Finite Difference Method, continued

- For these particular finite difference formulas, system to be solved is tridiagonal, which saves on both work and storage compared to general system of equations
- This is generally true of finite difference methods: they yield sparse systems because each equation involves few variables

Example: Convection-Diffusion Equation

$$-\nu \frac{d^2u}{dx^2} + c\frac{du}{dx} = 1, \quad u(0) = u(1) = 0,$$

Apply finite difference: $L\mathbf{u} = A\mathbf{u} + C\mathbf{u} = \mathbf{f}$

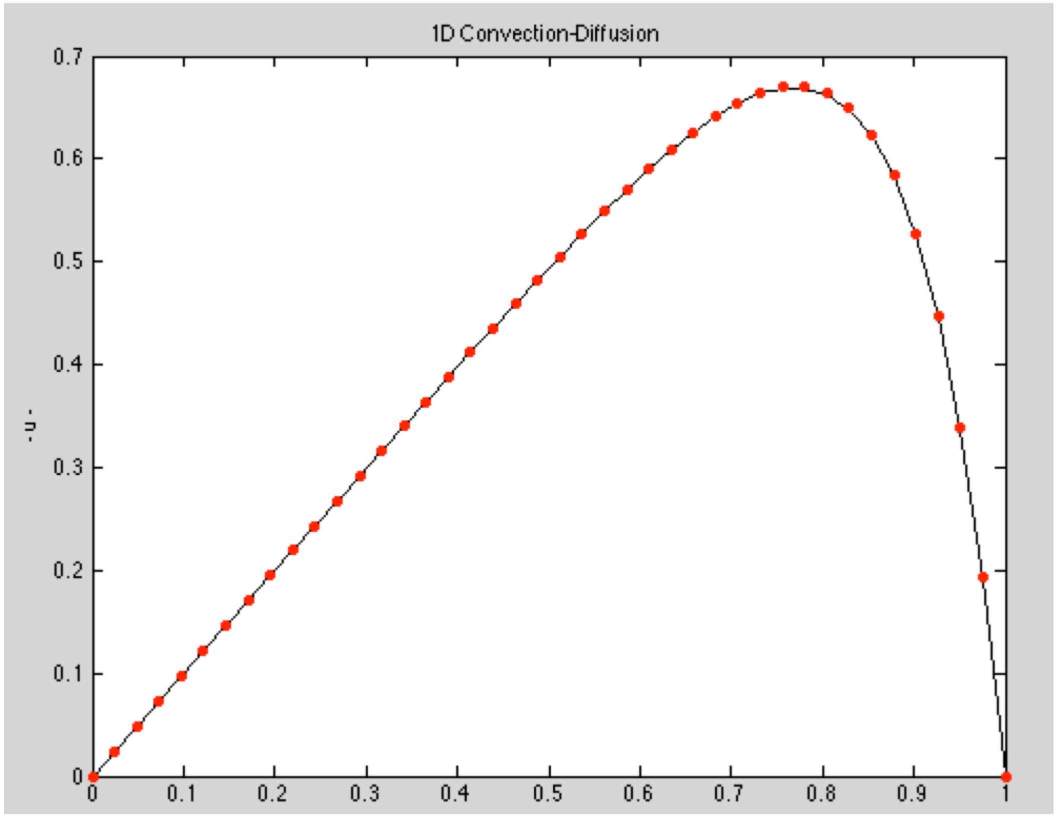
$$A = \frac{\nu}{\Delta x^2}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} \qquad C = \frac{c}{2\Delta x}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -1 & 0 \end{bmatrix}$$

MATLAB EXAMPLE

- A is symmetric positive definite.
- C is skew-symmetric.
- $L = A + C$ is neither SPD nor skew-symmetric.

Example: Convection-Diffusion Equation

```
format compact

a = 1; b=-2; c=1;
e = ones(n,1);
A = spdiags([a*e b*e c*e],-1:1, n,n);

a =-1; b=0; c=1;
e = ones(n,1);
C = spdiags([a*e b*e c*e],-1:1, n,n);

h = 1./(n+1);

nu = .1; c=0.001;
nu = .01; c=1;
nu = .001; c=1;

A = -(nu/(h*h))*A;
C = (c/(2*h))*C;

L = A+C;

f = ones(n,1);

u = L\f;

x = (1:n)*h; x=x';
x=[ 0; x ; 1];
u=[ 0; u ; 0];

ue = (x - (exp(c*x/nu)-1)/(exp(c/nu)-1) )/c;
ue = (x - ( exp(c*(x-1)/nu)-exp(-c/nu) )/(1-exp(-c/nu)) )/c;

plot(x,ue,'k-',x,u,'r.-')
title('1D Convection-Diffusion')
xlabel(' - x - ')
ylabel(' - u - ')

norm(ue-u)/norm(ue)
```



conv1d_a.m

Comments About Computing Error Norms

- Be careful with the l_2 vector norm!
- Even though $\max |e_i| \longrightarrow 0$ with $n \longrightarrow \infty$, we can still have $||\mathbf{e}||$ grow with n. Why?
- When solving differential equations, it is better to use norms that approximate their continuous counterparts. Thus

$$\begin{aligned} ||e||_2 &= \left[\int_\Omega e^2\,dx\right]^{\frac{1}{2}} \approx \left[\frac{1}{n}\sum_{i=1}^{n}|e_i|^2\right]^{\frac{1}{2}} \\ ||e||_\infty &= \max_\Omega |e| \approx \max_i |e_i| \end{aligned}$$

- The issue can also be resolved by measuring *relative error:*

$$\mathit{error} \;:=\; \frac{||\mathbf{e}||}{||\mathbf{u}||}$$

  for some appropriate vector norm.

- Still, best to start with a norm that doesn't scale with n.

Convection-Diffusion Equation

- Consider 1D convection-diffusion with $c = 1$ and $f = 1$:

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f$$

$$u(0) = 0, \quad u(1) = 0.$$

- Assume steady-state conditions $u_t = 0$

$$-\nu u_{xx} + c\,u_x = 1, \quad u(0) = u(1) = 0.$$

- If $\nu = 0$, we have:

$$c\,u_x = 1, \quad u(0) = u(1) = 0\ ???$$

Too many boundary conditions!

Convection-Diffusion Equation

- The issue is that $\nu \longrightarrow 0$ is a *singular perturbation*.
- This is true whenever the *highest-order derivative* is multiplied by a small constant.
- As the constant goes to zero, the number of boundary conditions changes.
- Here,
  - We go from one boundary condition when $\nu = 0$,
  - to two boundary conditions when $\nu > 0$ (even for $\nu \ll 1$).
- An example that is *not* a singular perturbation is

$$-u_{xx} + \epsilon\, u_x \;=\; 1, \quad u(0) \;=\; u(1) \;=\; 0, \qquad \epsilon \longrightarrow 0.$$

This is called a *regular perturbation*.

Regular / Singular Perturbations You're Familiar With

- Consider solutions to the quadratic equation: $ax^2 + bx + c = 0.$

Example 1: $x^2 + \epsilon x = 1:$ Two roots as $\epsilon \longrightarrow 0.$ *Regular perturbation.*

Example 2: $\epsilon x^2 + x = 1:$

$$\begin{aligned}
x &= -\frac{1}{2\epsilon} \pm \frac{1}{2\epsilon}\sqrt{1+4\epsilon} \\
x_1 &= \frac{1}{2\epsilon}\left(\sqrt{1+4\epsilon} - 1\right) \\
&= \frac{1}{2\epsilon}\left(1 + 2\epsilon - 1 + O(\epsilon^2)\right) \\
&= 1 + O(\epsilon). \\
x_2 &= -\frac{1}{2\epsilon}\left(2 + O(\epsilon)\right) \longrightarrow -\infty. \quad \textit{Singular perturbation.}
\end{aligned}$$

Convection-Diffusion Equation

- Exact solution for our 1D model problem:

$$\begin{aligned} u &= \frac{x}{c} - \frac{L}{c}\left[\frac{e^{cx/\nu} - 1}{e^{cL/\nu} - 1}\right] \\ &= \frac{1}{c}\left[x - \frac{e^{c(x-L)/\nu} - e^{-cL/\nu}}{1 - e^{-cL/\nu}}\right]. \end{aligned}$$

- In the convection-dominated limit $(cL \gg \nu)$, one of these is computable in IEEE floating point, one is not.

- Which is which?

Convection-Diffusion Equation

- What happens when cL/$\nu \gg 1$ in our numerical example?

Nonlinear Example: The Bratu Equation

- Consider 1D diffusion with nonlinear right-hand side:

$$-\frac{d^2u}{dx^2} \;=\; q(x,u) \;=\; \sigma\, e^u, \;\; u(0) = u(1) = 0.$$

- Discretizing with finite differences (say),

$$A\mathbf{u} = \sigma\, e^{\mathbf{u}}.$$

- Nonlinear system:

$$\mathbf{f}(\mathbf{u}) \;=\; 0, \quad \mathbf{f}(\mathbf{u}) \;=\; A\mathbf{u} \;-\; \sigma\, e^{\mathbf{u}}.$$

- Newton's method:

$$\begin{aligned} \mathbf{u}^{k+1} &= \mathbf{u}^k + \mathbf{s}^k \\ \mathbf{s}^k &= -[J^k]^{-1}\,\mathbf{f}(\mathbf{u}^k). \end{aligned}$$

$$\left(J^k\right)_{ij} := \frac{\partial f_i^k}{\partial u_j^k}.$$

- ith equation:

$$\mathbf{f}_i^k = \sum_{j=1}^{n} A_{ij}\,u_j^k - \sigma e^{u_i^k} \qquad \longrightarrow \qquad (J_k)_{ij} = \frac{\partial f_i}{\partial u_j} = A_{ij} - \sigma e^{u_i}.$$

- If $b = -1$ and $a_j = 2 - h^2 \sigma e^{u_j}$, then

$$J = \frac{1}{h^2} \begin{pmatrix} a_1 & b & & & \\ b & a_2 & b & & \\ & b & \ddots & \ddots & \\ & & \ddots & \ddots & b \\ & & & b & a_n \end{pmatrix},$$

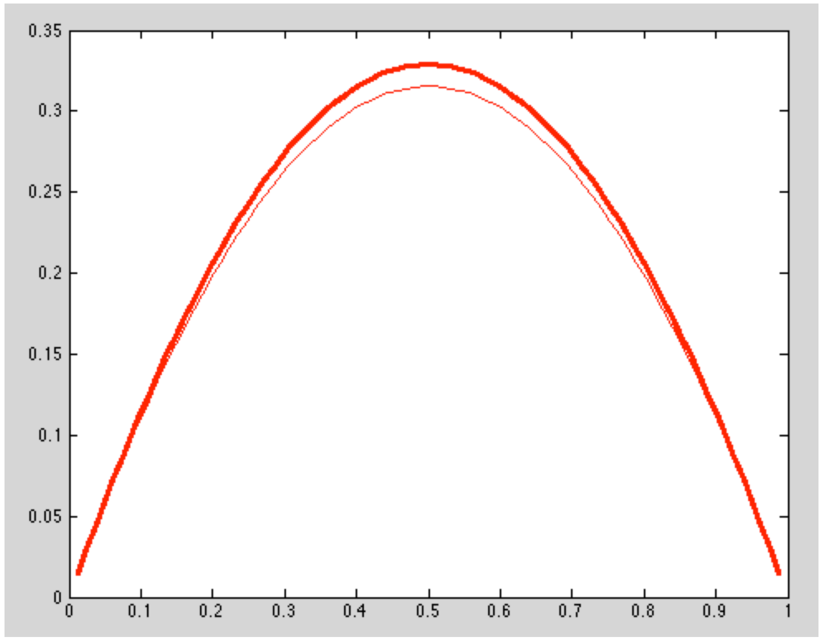
- At *each iteration*, modify the tridiagonal matrix A such that

$$J_k = A + \sigma e^{u_i^k} \delta_{ij},$$

and solve this tridiagonal system in $\approx 8n$ operations.

bratu1a.m / bratu_lin.m

```
format compact; format longe;  close all

n=80; sigma = 2;

h=1./(n+1); b = ones(n,1); x=1:n; x=h*x'; h2i = 1./(h*h);
hold off; plot(x,0*x,'k-'); hold on

a=-2*b;
A = h2i*spdiags([b a b],-1:1, n,n);

c=-2*b + sigma*h*h*exp(x); J = h2i*spdiags([b c b],-1:1, n,n);


u=b*0;

for iter=1:31;

   f=A*u + sigma*exp(u);
   c=-2*b + sigma*h*h*exp(u);
   J = h2i*spdiags([b c b],-1:1, n,n);

   [L,U]=lu(J);

   s = -U\ (L\f);

   u = u+s;

   plot(x,u,'r-'); hold on

   ns = norm(s); nf = norm(f); [ns nf]

end;

plot(x,u,'r-'); hold on
```



Extension of Finite Difference to Variable Coefficients

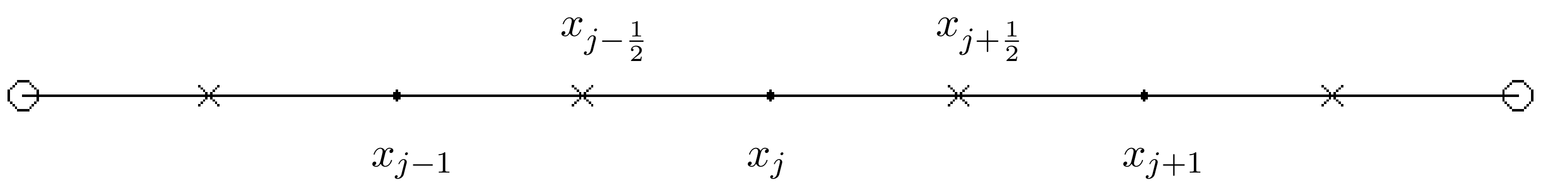


Figure 1: Grid spacing for variable coefficient diffusion operator.

Consider the one-dimesional model problem,

$$\frac{d}{dx}\,a(x)\frac{du}{dx} \;=\; f(x), \qquad u(0) \;=\; u(1) \;=\; 0. \tag{5}$$

Let

$$u_i \;:=\; u(x_i), \qquad a_{i+\frac{1}{2}} \;:=\; a(x_{i+\frac{1}{2}}). \tag{6}$$

with $x_i := i\,h$, $i = 0, \ldots, n+1$ and $x_{i+\frac{1}{2}} := (i+\frac{1}{2})\,h$, $i = 0, \ldots, n$, and $h := 1/(n+1)$. Then

$$w_i \;=\; \frac{d}{dx}\,a(x)\frac{du}{dx}\bigg|_{x_i} \;\approx\; \frac{1}{h}\left[\left(a\frac{du}{dx}\right)\bigg|_{x_{i+\frac{1}{2}}} \;-\; \left(a\frac{du}{dx}\right)\bigg|_{x_{i-\frac{1}{2}}}\right] \tag{7}$$

$$\approx\; \frac{1}{h}\left[a_{i+\frac{1}{2}}\left(\frac{u_{i+1}-u_i}{h}\right) \;-\; a_{i-\frac{1}{2}}\left(\frac{u_i-u_{i-1}}{h}\right)\right]. \tag{8}$$

Extension of Finite Difference to Variable Coefficients

Assuming $u = 0$ at the domain endpoints, then the finite difference appoximation to $u'_{i+\frac{1}{2}}$, $i = 0, \ldots, n$ can be evaluated as the matrix-vector product, $\underline{v} = D\underline{u}$, where D is the $(n+1) \times n$ finite difference matrix illustrated below.

$$\underline{v} \; = \; \begin{pmatrix} v_{\frac{1}{2}} \\ v_{\frac{3}{2}} \\ \vdots \\ v_{n+\frac{1}{2}} \end{pmatrix} \; = \; \frac{1}{h} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \; = \; D\underline{u}. \tag{9}$$

Note that $\frac{1}{h}(u_{i+1} - u_i)$ is generally regarded as a first-order accurate approximation to $\frac{du}{dx}$, it is in fact second-order accurate at the midpoint $x_{i+\frac{1}{2}}$.

Given $v_{i+\frac{1}{2}}$, it remains to evaluate the outer finite difference in (7), which maps data from the $(n+1)$ half-points to the n integer points. Let

$$q_{i+\frac{1}{2}} \quad := \quad a_{i+\frac{1}{2}} v_{i+\frac{1}{2}}. \tag{10}$$

Then

$$\underline{w} \; = \; \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} \; = \; \frac{1}{h} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ & & & -1 & 1 \end{bmatrix} \begin{pmatrix} q_{\frac{1}{2}} \\ q_{\frac{3}{2}} \\ \vdots \\ q_{n+\frac{1}{2}} \end{pmatrix} \; = \; -D^T\underline{q}. \tag{11}$$

Extension of Finite Difference to Variable Coefficients

Finally, note that if A is an $(n+1) \times (n+1)$ diagonal matrix with entries $(a_{\frac{1}{2}},\, a_{\frac{3}{2}}, \ldots a_{n+\frac{1}{2}})$, then (10) can be expressed as $\underline{q} = A\underline{v}$, and the finite-difference approximation (7) can be compactly expressed in matrix form as

$$\underline{w} = -D^T A D \underline{u} \qquad (12)$$

Assuming $a_{i+\frac{1}{2}} > 0$, it is easy to show that the matrix

$$L \;\; := \;\; D^T A D \qquad (13)$$

is symmetric positive definite, which is a requirement if the system is to be solved with conjugate gradient iteration or Cholesky factorization. Fortunately, this property carries over into the multidimensional case, which we consider in the next section. We further remark that L is a map from data $(u_1 \ldots u_j \ldots u_n)$ to $(w_1 \ldots w_j \ldots w_n)$. That is, once defined, it does not generate data at the half gridpoint locations. This is a particularly attractive feature in multiple space dimensions where having multiple grids leads to an explosion of notational difficulties.

Convergence Behavior: Finite Difference

- In differential equations, we are interested in the rate of convergence – i.e., the rate at which the error goes to zero vs. n, the number of unknowns in the system.

- For finite difference methods and methods using Lagrangian interpolants, n is the number of gridpoints (but, depends on the type of boundary conditions…..)

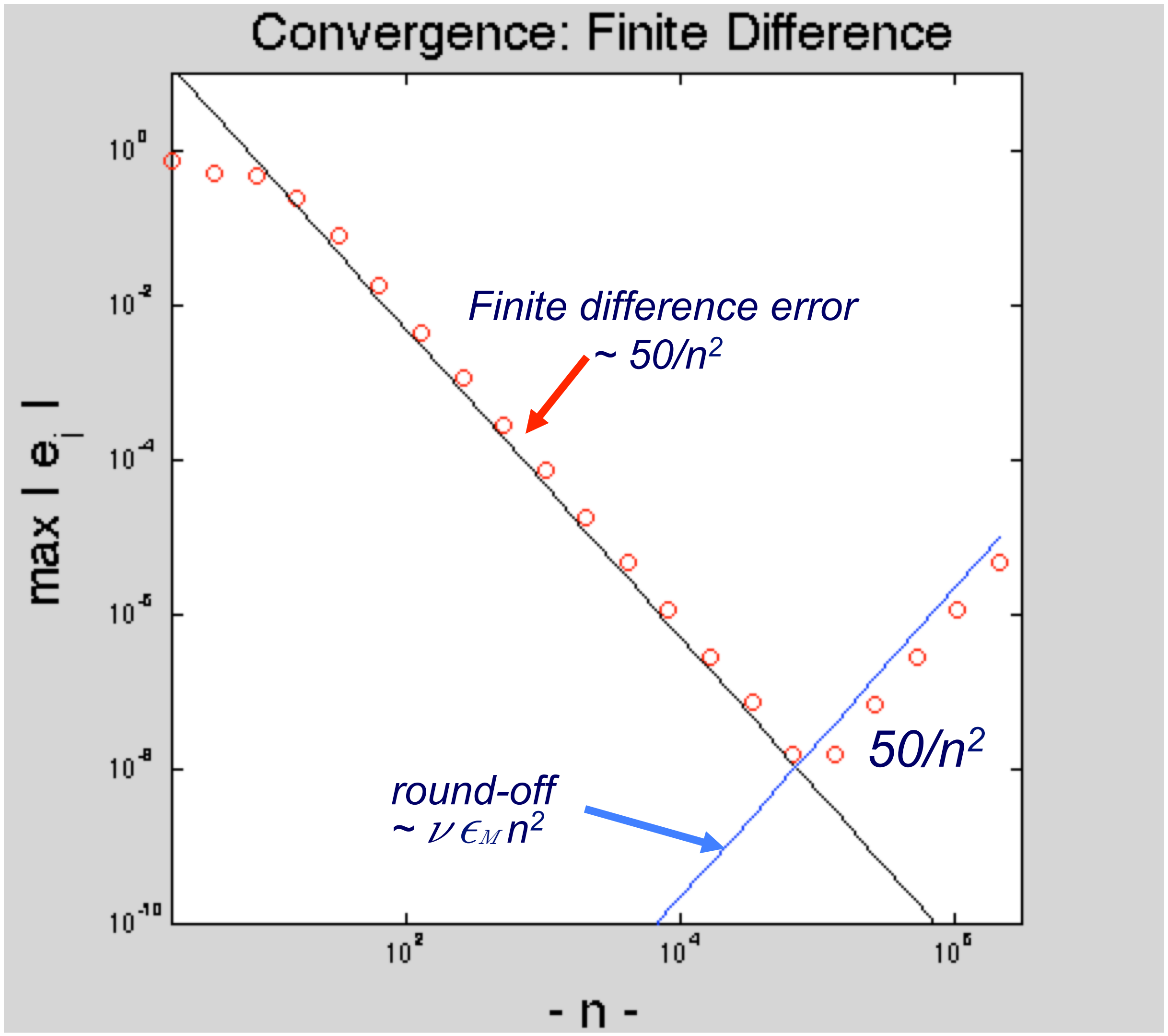
- The next figure shows the error vs. n for a 2nd-order (i.e., $O(h^2)$) finite difference solution of the steady-state convection-diffusion equation in 1D.

- For $n > \sim \epsilon_M^{-1/3}$, the error goes up, due to round-off associated with the approximation to the 2nd-order derivative.

- As we've seen in past homework assignments, the minimum error is around $\epsilon_M^{-1/2}$

Finite Difference Convergence Rate

conv1d.m



Properties of Finite Difference Methods

- Pros
  - Easy to formulate (for simple problems)
  - Easy to analyze "
  - Easy to code "
  - Closed-form expressions for eigenvalues/eigenvectors for uniform grid with constant coefficients.

- Cons –
  - Geometric complexity for 2D/3D is not as readily handled as FEM.
  - Difficult to extend to high-order (because of boundary conditions).
  - Do not always (e.g., because of BCs) get a symmetric matrix for

$$\frac{d^2u}{dx^2} \qquad \text{or} \qquad \frac{d}{dx}\nu(x)\frac{du}{dx}$$

Eigenvalues, Continuous and Discrete

- One of the great features of finite difference methods is that one can readily compute the eigenvalues of the discrete operators and thus understand their spectrum and convergence rates.

- The latter is important for understanding accuracy.

- The former is important for understanding stability of time-stepping schemes in the case of PDEs, which we'll see in the next chapter.

- The reason it is easy to find the eigenvalues for finite difference methods is that, for the constant coefficient case, they often share the same eigenfunctions as their continuous counterparts.

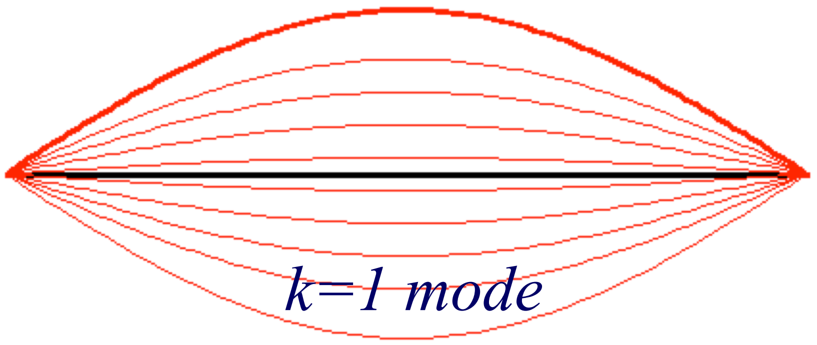
Eigenvalue Example:

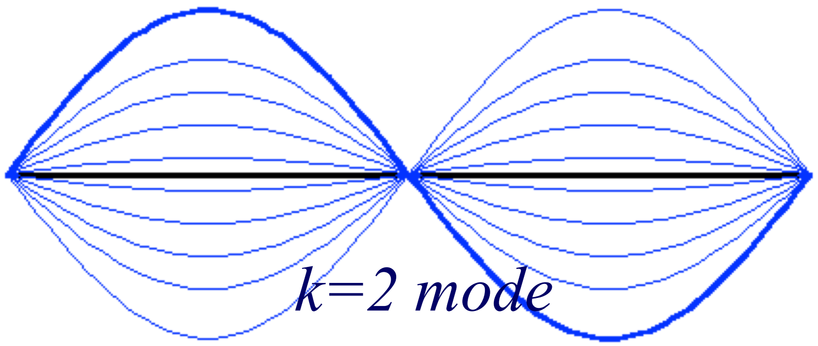
- Consider the analytical (i.e., continuous) eigenvalue problem

$$-\frac{d^2\tilde{u}}{dx^2} = \tilde{\lambda}\,\tilde{u}, \qquad \tilde{u}(0) = \tilde{u}(1) = 0.$$

- The eigenfunctions/eigenvalues for the continuous problem are

$$\begin{aligned} \tilde{u} &= \sin(k\pi x) : \\ -\tilde{u}'' &= k^2\pi^2 \sin(k\pi x) \;=\; k^2\pi^2\,\tilde{u} \;=\; \tilde{\lambda}_k\,\tilde{u} \\ \tilde{\lambda}_k &= k^2\pi^2 \end{aligned}$$





The modes are like the vibrations of a guitar string.
Higher wavenumbers, k, correspond to higher frequency.
Here, the k=2 mode would be a harmonic – one octave higher.

Finite Difference Eigenvectors/values:

- Consider $\mathbf{s} = [\sin(k\pi x_j)]^T$:

$$\begin{aligned} A\mathbf{s}|_j &= \frac{-1}{\Delta x^2}\left[\sin k\pi x_{j+1} \;-\; 2\sin k\pi x_j \;+\; \sin k\pi x_{j-1}\right] \\ &= \frac{-1}{\Delta x^2}\left[\sin(k\pi x_j + \Delta x) \;-\; 2\sin(k\pi x_j) \;+\; \sin(k\pi x_j - \Delta x)\right] \end{aligned}$$

- Use the identity:

$$\sin(a+b) \;=\; \sin a\,\cos b \;+\; \cos a\,\sin b$$

$$\begin{aligned} \sin(k\pi x_{j+1}) &= \sin k\pi x_j\;\cos k\pi\Delta x \;+\; \cos k\pi x_j\;\sin k\pi\Delta x \\ \sin(k\pi x_{j-1}) &= \sin k\pi x_j\;\cos k\pi\Delta x \;-\; \cos k\pi x_j\;\sin k\pi\Delta x \\ \text{SUM} &= 2\;\sin k\pi x_j\;\cos k\pi\Delta x \end{aligned}$$

- $$\begin{aligned} A\mathbf{s}|_j = \frac{-1}{\Delta x^2}\left[s_{j+1} \;-\; 2s_j \;+\; s_{j-1}\right] &= -\frac{1}{\Delta x^2}\left[2\cos k\pi\Delta x - 2\right]\sin k\pi x_j \\ &= \lambda_k \mathbf{s}|_j \end{aligned}$$

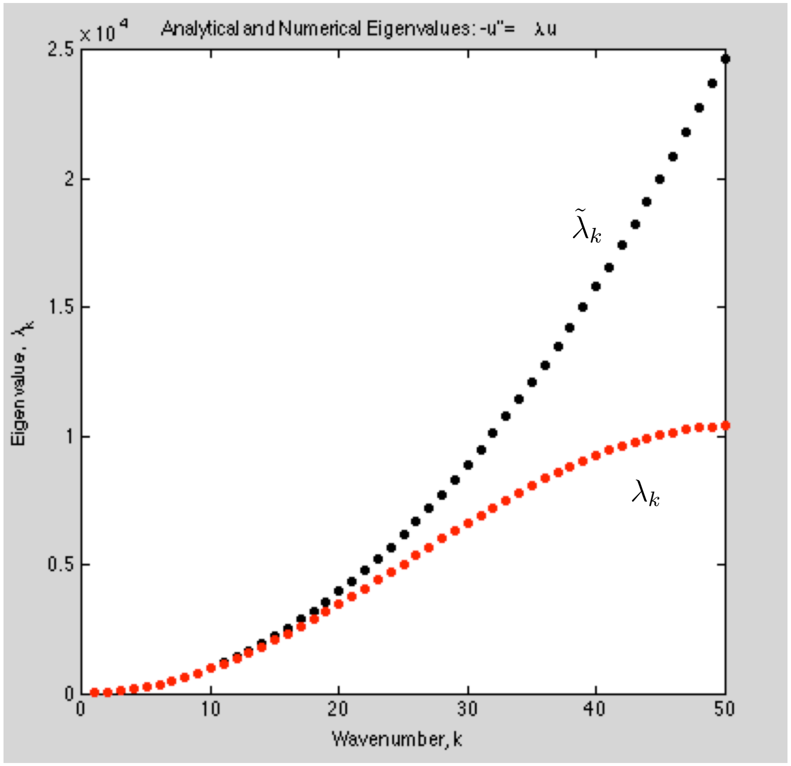
$$\boxed{\lambda_k \;=\; \frac{2}{\Delta x^2}\left[1 - \cos k\pi\Delta x\right].}$$

Eigenvalue Properties for $-u'' = \lambda u$, $u(0) = u(1) = 0$:

- $\max \lambda_k \sim Cn^2$ $\qquad \dfrac{\tilde{\lambda}_n}{\lambda_n} \sim \dfrac{\pi^2}{4}$

- For $k\Delta x \ll 1$, (with $\theta := k\Delta x$):

$$\lambda_k = (k\pi)^2 \left[1 - \frac{(k\pi\Delta x)^2}{12} + \cdots \right]$$



Collocation Method

- *Collocation method* approximates solution to BVP by finite linear combination of basis functions
- For two-point BVP

$$u'' = f(t, u, u'), \qquad a < t < b$$

with BC

$$u(a) = \alpha, \qquad u(b) = \beta$$

we seek approximate solution of form

$$u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$$

where ϕ_i are basis functions defined on $[a, b]$ and $\boldsymbol{x}$ is n-vector of parameters to be determined

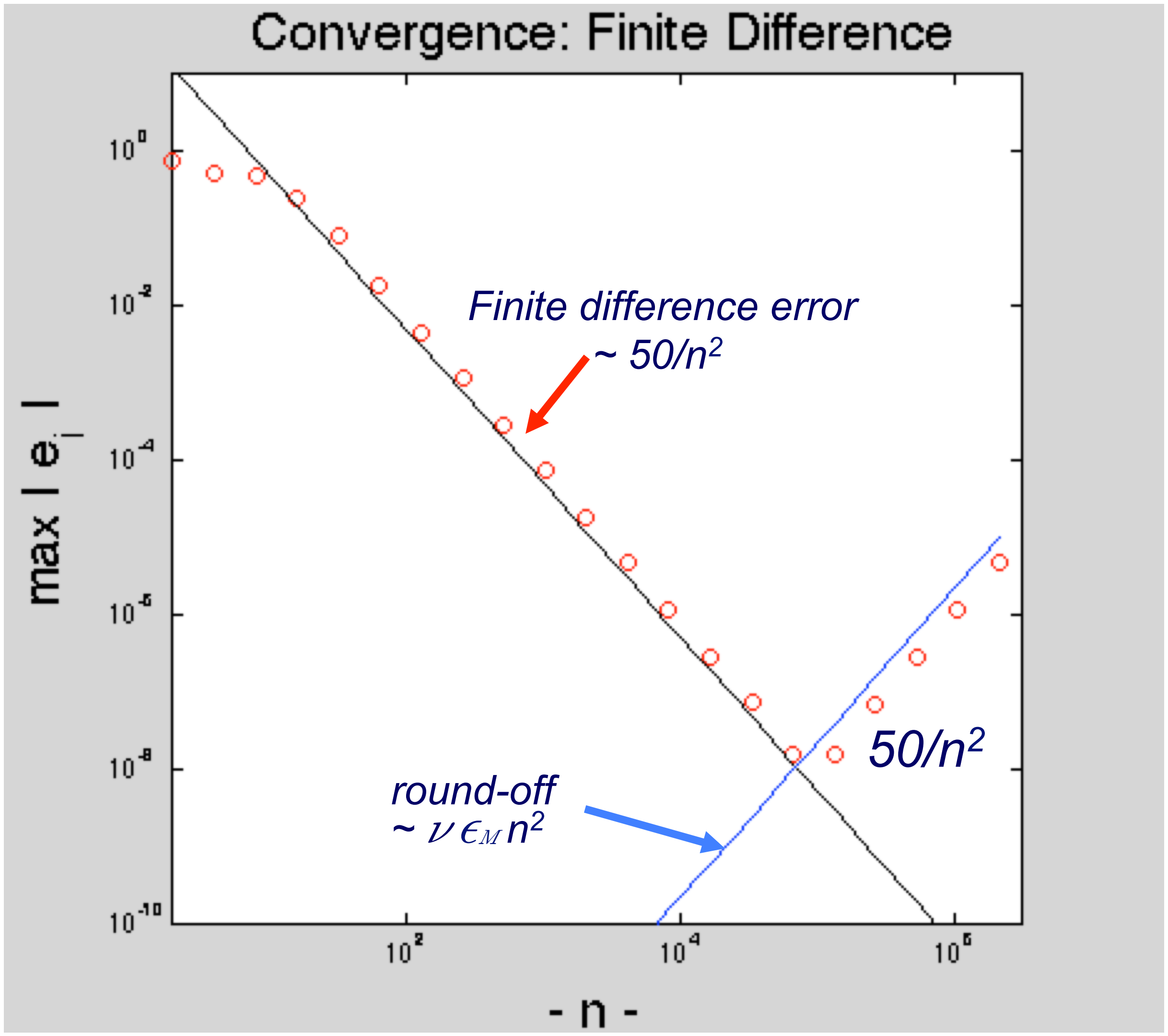
Collocation Method, continued

- To determine vector of parameters $\boldsymbol{x}$, define set of n *collocation points*, $a = t_1 < \cdots < t_n = b$, at which approximate solution $v(t, \boldsymbol{x})$ is forced to satisfy ODE and boundary conditions
- Common choices of collocation points include equally-spaced points or Chebyshev points
- Suitably smooth basis functions can be differentiated analytically, so that approximate solution and its derivatives can be substituted into ODE and BC to obtain system of algebraic equations for unknown parameters $\boldsymbol{x}$

Collocation

- Collocation is essentially the ***method of undetermined coefficients***.
  - Start with
$$u(x) = \sum_{j=0}^{n} \hat{u}_j \phi_j(x)$$
.
  - Find coefficients $\hat{u}_j$ such that BVP is satisfied at gridpoints x_i.
- Instead of using monomials, $\phi_j = x^j$, could use Lagrange polynomials on Chebyshev or Legendre quadrature points.
- Normally, one would use Gauss-Lobatto-Legendre or Gauss-Lobatto-Chebyshev points, which include ± 1 (i.e., the endpoints of the interval) in the nodal set.
- If the solution is to be zero at those boundaries one would have $u_0 = u_n = 0$.
- In many cases, these methods are exponentially convergent. (**Counter-example:** sines and cosines, unless problem is periodic.)
- For several reasons conditioning, symmetry, robustness, and ease of boundary condtions, collocation has lost favor to Galerkin methods.

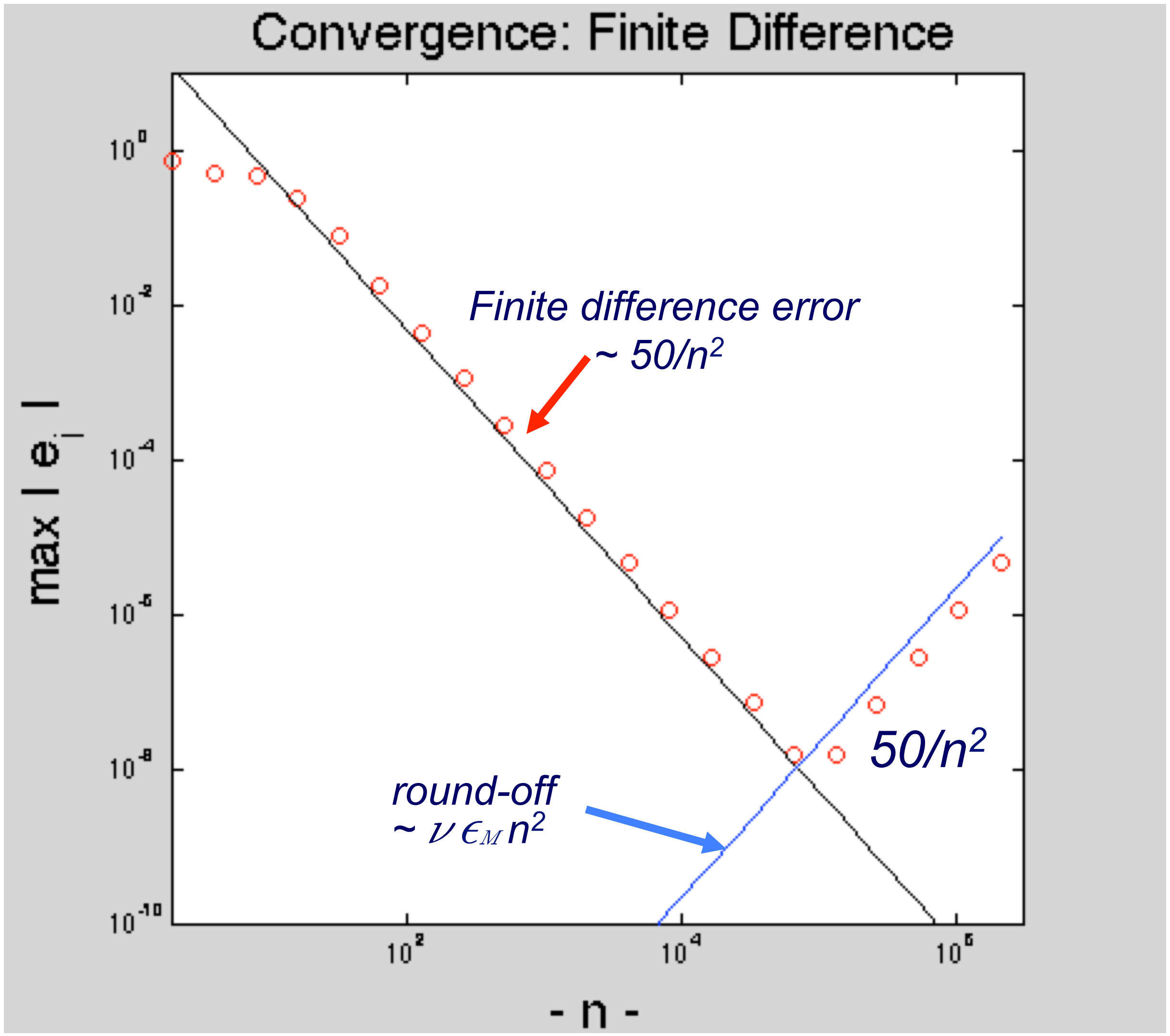
Finite Difference Convergence Rate



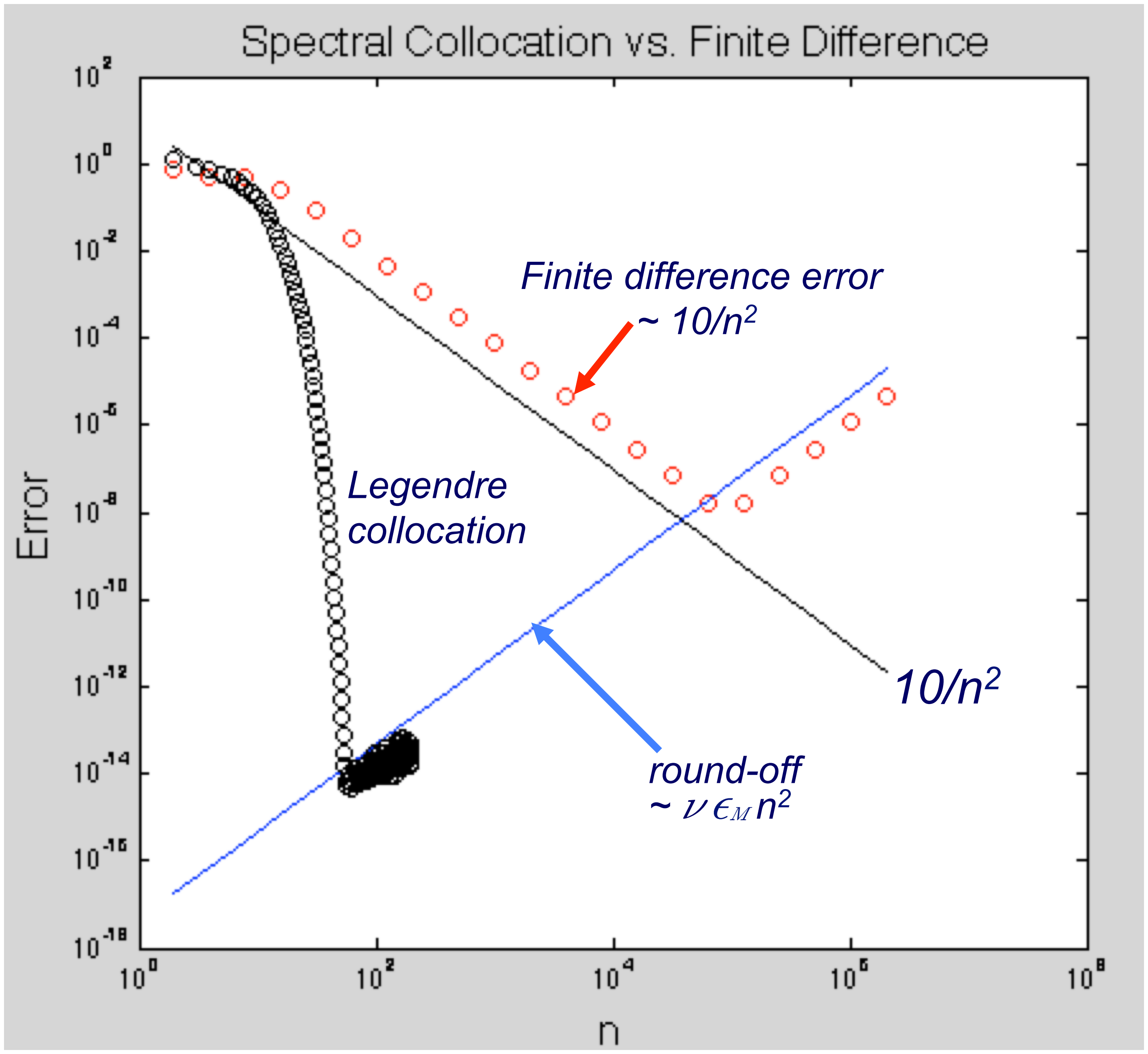
Convergence Behavior: High-Order Methods

- The 2nd-order convergence of standard finite difference methods looks reasonable.
- However, higher-order methods are generally much faster in that the same error can be achieved with a lower **n**, once **n** is large enough for the asymptotic convergence behavior to apply.
- High-order methods suffer the same round-off issue, with error growing like $\epsilon_M \mathbf{n}^2$.
- However, their truncation error goes to zero more rapidly so that the value of **n** where truncation and round-off errors balance is lower. The minimum error is thus much smaller for high-order methods.
- Usually, we are more interested in a small error at small **n**, rather than realizing the minimum possible error.
- For PDEs on an (**n** x **n** x **n**) grid cost generally scales as $\mathbf{n}^3$, so a smaller **n** is a significant win.

Revisiting our finite difference result, we contrast the second-order convergence with nth-order collocation on the next slide.

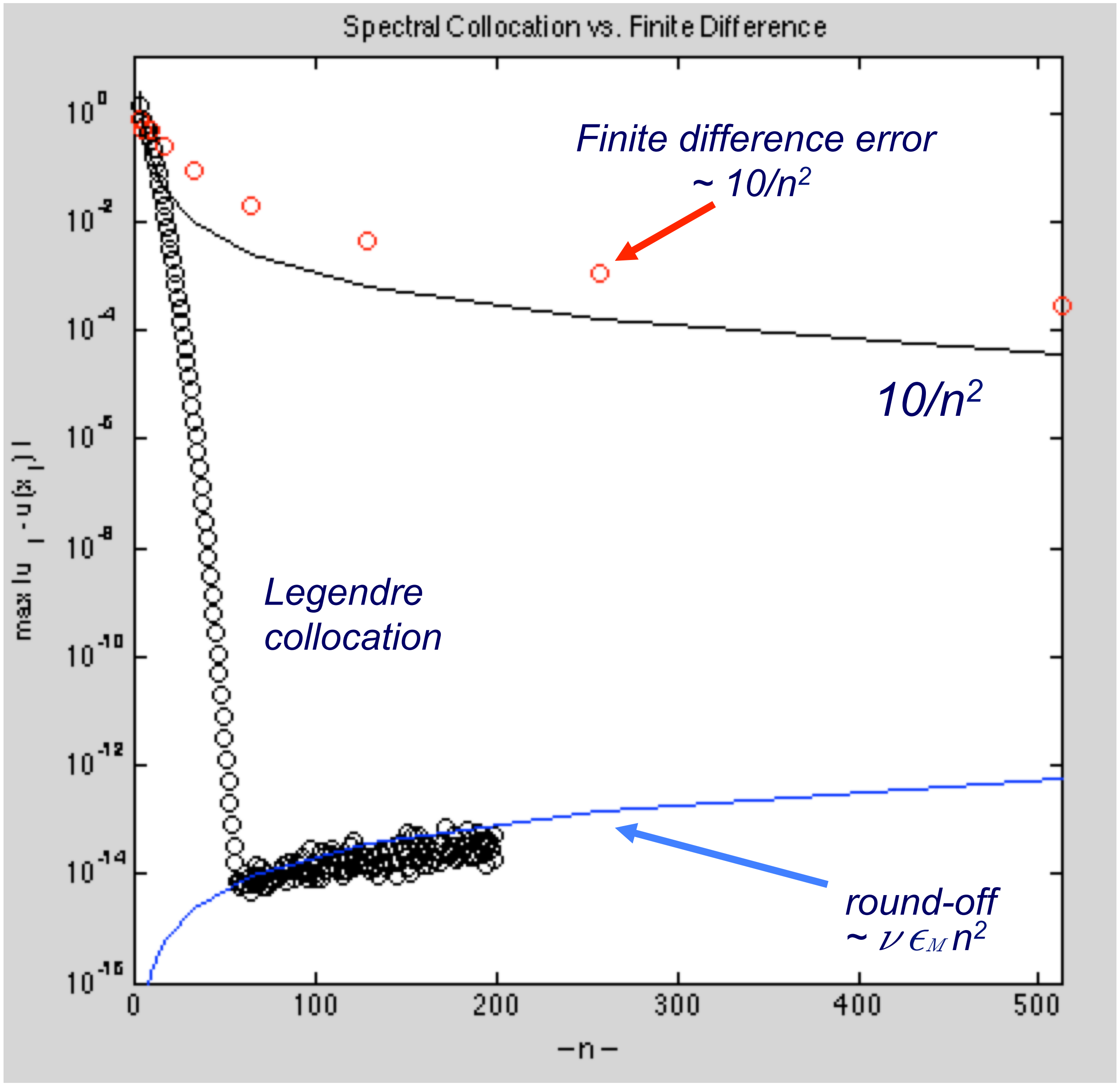


Spectral Collocation vs. Finite Difference



conv1d.m
collocation.m
plotcol.m

Spectral Collocation vs. Finite Difference (semilogy)



Galerkin Method

- Rather than forcing residual to be zero at finite number of points, as in collocation, we could instead minimize residual over entire interval of integration
- For example, for *Poisson equation* in one dimension,

$$u'' = f(t), \qquad a < t < b,$$

with homogeneous BC $u(a) = 0, \qquad u(b) = 0,$

subsitute approx solution $u(t) \approx v(t, \boldsymbol{x}) = \sum_{i=1}^{n} x_i \phi_i(t)$

into ODE and define residual

$$r(t, \boldsymbol{x}) = v''(t, \boldsymbol{x}) - f(t) = \sum_{i=1}^{n} x_i \phi_i''(t) - f(t)$$

Galerkin Method, continued

- More generally, *weighted residual method* forces residual to be orthogonal to each of set of *weight functions* or *test functions* w_i,

$$\int_a^b r(t, \boldsymbol{x}) w_i(t)\, dt = 0, \quad i = 1, \ldots, n$$

- This yields linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where now

$$a_{ij} = \int_a^b \phi_j''(t) w_i(t)\, dt, \qquad b_i = \int_a^b f(t) w_i(t)\, dt$$

whose solution gives vector of parameters $\boldsymbol{x}$

Galerkin Method, continued

- Matrix resulting from weighted residual method is generally not symmetric, and its entries involve second derivatives of basis functions
- Both drawbacks are overcome by *Galerkin method*, in which weight functions are chosen to be *same* as basis functions, i.e., $w_i = \phi_i$, $i = 1, \ldots, n$
- Orthogonality condition then becomes

$$\int_a^b r(t, \boldsymbol{x})\phi_i(t)\, dt = 0, \quad i = 1, \ldots, n$$

or

$$\int_a^b v''(t, \boldsymbol{x})\phi_i(t)\, dt = \int_a^b f(t)\phi_i(t)\, dt, \quad i = 1, \ldots, n$$

Galerkin Method, continued

- Degree of differentiability required can be reduced using integration by parts, which gives

$$\begin{aligned}\int_a^b v''(t,\boldsymbol{x})\phi_i(t)\,dt &= v'(t)\phi_i(t)\big|_a^b - \int_a^b v'(t)\phi_i'(t)\,dt \\ &= v'(b)\phi_i(b) - v'(a)\phi_i(a) - \int_a^b v'(t)\phi_i'(t)\,dt\end{aligned}$$

- Assuming basis functions ϕ_i satisfy homogeneous boundary conditions, so $\phi_i(0) = \phi_i(1) = 0$, orthogonality condition then becomes

$$-\int_a^b v'(t)\phi_i'(t)\,dt = \int_a^b f(t)\phi_i(t)\,dt, \quad i = 1,\ldots,n$$

Galerkin Method, continued

- This yields system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$, with

$$a_{ij} = -\int_a^b \phi_j'(t)\phi_i'(t)\,dt, \qquad b_i = \int_a^b f(t)\phi_i(t)\,dt$$

  whose solution gives vector of parameters $\boldsymbol{x}$

- $\boldsymbol{A}$ is symmetric and involves only first derivatives of basis functions

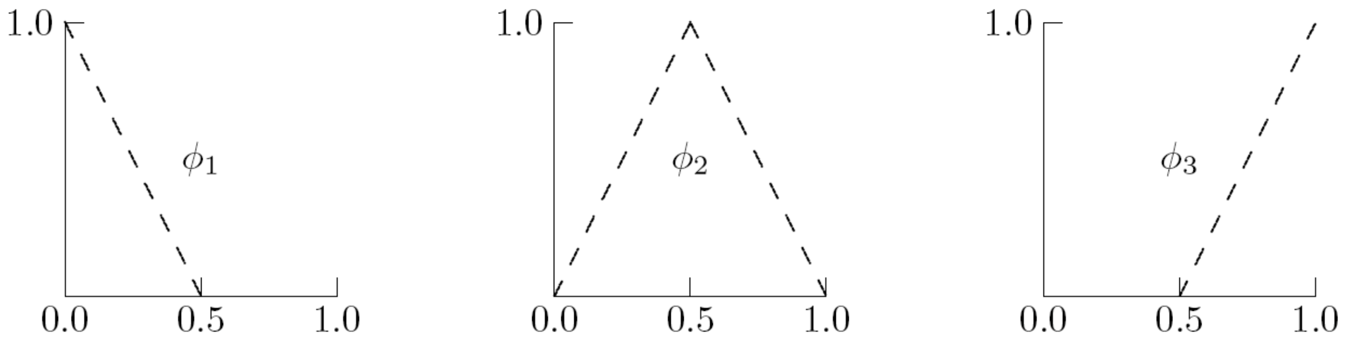
Example: Galerkin Method

- Consider again two-point BVP

$$u'' = 6t, \qquad 0 < t < 1,$$

with BC $\quad u(0) = 0, \qquad u(1) = 1$

- We will approximate solution by piecewise linear polynomial, for which B-splines of degree 1 (“hat” functions) form suitable set of basis functions



- To keep computation to minimum, we again use same three mesh points, but now they become knots in piecewise linear polynomial approximation

Example, continued

- Thus, we seek approximate solution of form

$$u(t) \approx v(t, \boldsymbol{x}) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t)$$

- From BC, we must have $x_1 = 0$ and $x_3 = 1$
- To determine remaining parameter x_2, we impose Galerkin orthogonality condition on interior basis function ϕ_2 and obtain equation

$$-\sum_{j=1}^{3}\left(\int_0^1 \phi_j'(t)\phi_2'(t)\,dt\right)x_j = \int_0^1 6t\phi_2(t)\,dt$$

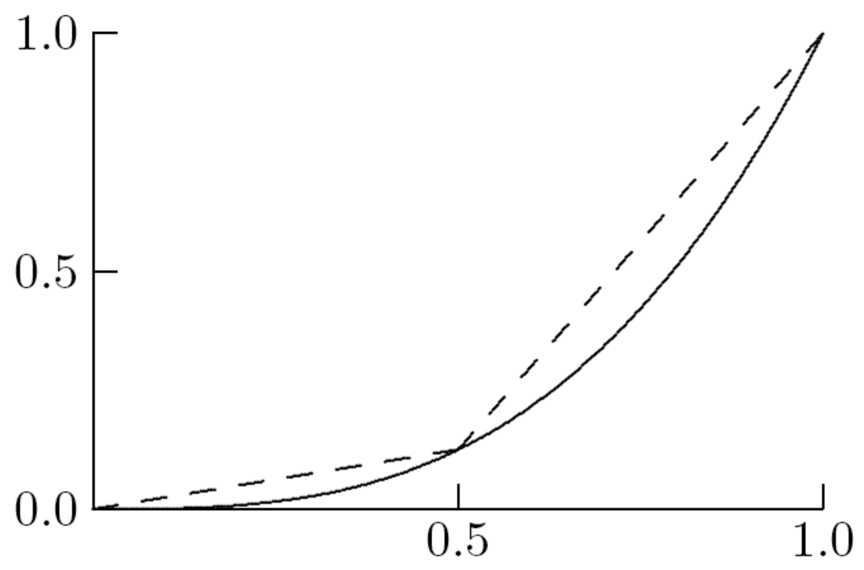
or, upon evaluating these simple integrals analytically

$$2x_1 - 4x_2 + 2x_3 = 3/2$$

Example, continued

- Substituting known values for x_1 and x_3 then gives $x_2 = 1/8$ for remaining unknown parameter, so piecewise linear approximate solution is

$$u(t) \approx v(t, \boldsymbol{x}) = 0.125\phi_2(t) + \phi_3(t)$$



- We note that $v(0.5, \boldsymbol{x}) = 0.125$

Example, continued

- More realistic problem would have many more interior mesh points and basis functions, and correspondingly many parameters to be determined
- Resulting system of equations would be much larger but still sparse, and therefore relatively easy to solve, provided local basis functions, such as “hat” functions, are used
- Resulting approximate solution function is less smooth than true solution, but nevertheless becomes more accurate as more mesh points are used

< interactive example >

Weighted Residual Example: $-\dfrac{d^2\tilde{u}}{dx^2} = f(x),\ \tilde{u}(0) = \tilde{u}(1) = 0.$

- Trial function (*n unknowns.*)

$$u(x) \;=\; \sum_{j=1}^{n} u_j\,\phi_j(x) \;\in X_0^N : \quad \phi_j(0) \;=\; \phi_j(1) \;=\; 0,\ j = 1,\ldots,n.$$

- Residual (w.r.t. u): $\qquad r(u) \;:=\; f \;+\; \dfrac{d^2u}{dx^2}$

- Orthogonality condition:

$$(v, r) \;:=\; \int_0^1 v\,r\,dx \;=\; 0 \;\;\forall\, v \;\in\; X_0^N.$$

- Orthogonality condition:

$$(v, r) \;:=\; \int_0^1 v\, r\, dx \;=\; 0 \;\; \forall\, v \,\in\, X_0^N .$$

- So, for $i = 1, \dots, n$, $v = \phi_i$ (n *equations*):

$$(\phi_i, r) \;:=\; \int_0^1 \phi_i\, r\, dx \;=\; 0 \;=\; \int_0^1 \phi_i \left(f \,+\, \frac{d^2u}{dx^2} \right) dx$$

- Or,

$$-\int_0^1 \phi_i \, \frac{d^2u}{dx^2}\, dx \;=\; \int_0^1 \phi_i \, f \, dx .$$

- An important refinement:

$$-\int_0^1 \phi_i \, \frac{d^2u}{dx^2} \, dx \;=\; \int_0^1 \phi_i \, f \, dx.$$

- We can avoid requiring existence of u'' by interpolating the expression on the left by parts:

$$\mathcal{I} \;:=\; -\int_0^1 \underbrace{\phi_i}_{\text{``}v\text{''}} \; \underbrace{\frac{d^2u}{dx^2}}_{\text{``}du\text{''}} \; dx \;=\; \int_0^1 \frac{du}{dx}\frac{d\phi_i}{dx} \, dx \;-\; \phi_i \, \frac{du}{dx}\bigg|_{x=0}^{x=1}$$

$$= \int_0^1 \frac{d\phi_i}{dx}\frac{du}{dx} \, dx \qquad (\phi_i(0) = \phi_i(1) = 0 \text{ for these boundary conditions}).$$

- Now the test and trial functions require only the existence of a first derivative. (Admits a larger search space for approximate solutions.)

- Inserting the expansion for $\dfrac{du}{dx}$:

$$\int_0^1 \frac{d\phi_i}{dx}\frac{du}{dx}\,dx \;=\; \int_0^1 \frac{d\phi_i}{dx}\left(\sum_{j=1}^{n} u_j \frac{d\phi_j}{dx}\right)\,dx \;=\; \int_0^1 \phi_i(x)\,f(x)\,dx \;=:\; b_i.$$

$$\sum_{j=1}^{n} \underbrace{\left(\int_0^1 \frac{d\phi_i}{dx}\,\frac{d\phi_j}{dx}\,dx\right)}_{=:a_{ij}} u_j \;=\; b_i.$$

- In matrix form:

$$A\underline{u} \;=\; \underline{b} \quad \left\{ \begin{array}{lllll} a_{ij} & := \displaystyle\int_0^1 \frac{d\phi_i}{dx}\,\frac{d\phi_j}{dx}\,dx & = & (\dfrac{d\phi_i}{dx}, \dfrac{d\phi_j}{dx}) & =: \; a(\phi_i, \phi_j). \\ b_i & := \displaystyle\int_0^1 \phi_i\,f\,dx & = & (\phi_i, f) & \end{array} \right.$$

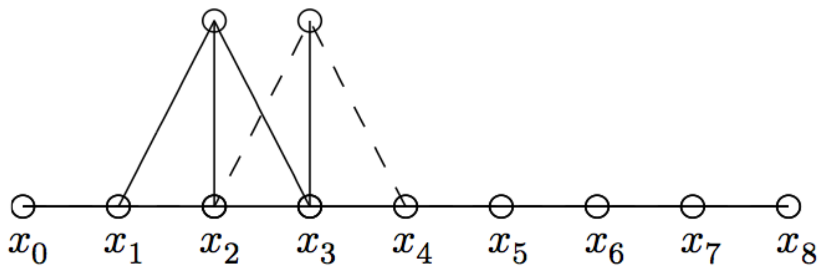
- We refer to $a(\phi_i, \phi_j)$ as the *a inner product.*

Example: Linear Finite Elements in 1D

- Piecewise-linear basis functions, $\phi_i(x)$, $i = 1, \ldots, n$, $x \in \Omega = [0, 1]$:

$$
\begin{aligned}
\phi_i(x) &= \frac{x - x_{i-1}}{x_i - x_{i-1}} \quad x \in [x_{i-1}, x_i] \\
&= \frac{x_{i+1} - x}{x_{i+1} - x_i} \quad x \in [x_i, x_{i+1}] \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\phi_i}{dx} &= \frac{1}{h_i} \quad x \in [x_{i-1}, x_i] \\
&= \frac{1}{h_{i+1}} \quad x \in [x_i, x_{i+1}] \\
&= 0, \quad \text{otherwise.}
\end{aligned}
$$



- Stiffness matrix entries:

$$\begin{aligned}
a_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx &= \frac{h_i}{h_i^2} + \frac{h_{i+1}}{h_{i+1}^2} \qquad j = i, \\
&= -\frac{h_i}{h_i^2} \qquad j = i-1, \\
&= -\frac{h_{i+1}}{h_{i+1}^2} \qquad j = i+1, \\
&= 0, \quad \text{otherwise.}
\end{aligned}$$

- For uniform gridspacing h,

$$a_{ij} = \int_0^1 \frac{d\phi_i}{dx}\frac{d\phi_j}{dx}\,dx = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}.$$

Note Limits!

- For the right-hand side,

$$b_i = \int_0^1 \phi_i \, f \, dx \approx \int \phi_i \left(\sum_{j=0}^{n+1} \phi_j(x) f_j \right) dx$$

$$= \sum_{j=0}^{n+1} b_{ij} f_j \, dx = \left(\bar{B} \underline{f} \right)_i .$$

- $\bar{B}$ is rectangular (mass matrix):

$$\left(\bar{B}\right)_{ij} := \int_0^1 \phi_i \, \phi_j \, dx = \frac{1}{3}(h_i + h_j) \qquad \text{if } j = i$$

$$= \frac{1}{6} h_i \qquad \text{if } j = i-1$$

$$= \frac{1}{6} h_{i+1} \qquad \text{if } j = i+1$$

- If $h = \frac{1}{n+1}$ is uniform, the $n \times (n+2)$ matrix $\bar{B}$ is,

$$\bar{B} \;=\; h \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix}.$$

The system to be solved is then

$$A\underline{u} \;=\; \bar{B}\underline{f}.$$

- Example: $$-\frac{d^2\tilde{u}}{dx^2} = 1, \quad u(0) = u(1) = 0.$$

- Exact solution: $\tilde{u} = \frac{1}{2}x(1-x).$

$$\boxed{\tilde{u}(x=1/3) = \frac{1}{2}\cdot\frac{1}{3}\left(1-\frac{1}{3}\right) = \frac{1}{9}.}$$

- FEM solution with $n=2$, $h=\frac{1}{3}$,

$$A\underline{u} = \frac{1}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = h\begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= h\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Solving for $\underline{u}$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \cdot h^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- Recall,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

- So,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{9} \cdot \frac{1}{3} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{9} \\ \frac{1}{9} \end{pmatrix}.$$

Extensions:

- Variable coefficients
- Neuman boundary conditions
- High-order polynomial bases
- *Best-fit property*

Variable Coefficients

- Consider $k(x) > 0$ for $x \in \Omega := [0,1]$,

$$-\frac{d}{dx}\, k\, \frac{d\tilde{u}}{dx} \;=\; f, \quad \tilde{u}(0) = \tilde{u}(1) = 0.$$

- WRT: *find $u \in X_0^N$ such that*

$$-\int_\Omega v \left(\frac{d}{dx}\, k\, \frac{du}{dx} \right) dx \;=\; \int_\Omega v\, f\, dx \quad \forall\, v \,\in X_0^N.$$

- Integrate by parts

$$a(v,u) \;:=\; \int_\Omega \frac{dv}{dx}\, k\, \frac{du}{dx}\, dx \;=\; \int_\Omega v\, f\, dx.$$

- Set $v = \phi_i(x)$, $u = \sum_j \phi_j(x)\, u_j$,

$$A\underline{u} \;=\; \bar{B}\underline{f}, \qquad a_{ij} \;:=\; \int_0^1 \frac{d\phi_i}{dx}\, k\, \frac{d\phi_j}{dx}\, dx.$$

- Becasuse $k > 0$, A is SPD. (!)

Neumann Boundary Conditions

- Consider $k(x) > 0$ for $x \in \Omega := [0,1]$,

$$-\frac{d}{dx}\, k\, \frac{d\tilde{u}}{dx} \;=\; f, \quad \tilde{u}(0) = 0,\; \left.\frac{du}{dx}\right|_{x=1} = 0.$$

- Now, include $\phi_{n+1}(x)$ as a valid basis function:

$$u \;=\; \sum_{j=1}^{n+1} \phi_j(x) u_j, \quad v = \phi_i(x), \quad i = 1, \dots, n+1.$$

- Integration by parts

$$\begin{aligned} -\int_\Omega v \left(\frac{d}{dx}\, k\, \frac{du}{dx} \right) dx \;&=\; \int_\Omega \frac{dv}{dx}\, k\, \frac{du}{dx}\, dx \;-\; \left(\cancel{v \left.\frac{du}{dx}\right|_{x=1} -} \; v \left.\frac{du}{dx}\right|_{x=0} \right) \\ &=\; \int_\Omega \frac{dv}{dx}\, k\, \frac{du}{dx}\, dx \;=\; \int_\Omega v\, f\, dx. \end{aligned}$$

- Set $v = \phi_i(x)$, $u = \sum_j \phi_j(x)\, u_j$,

$$A\underline{u} \;=\; \bar{B}\underline{f}, \qquad a_{ij} \;:=\; \int_0^1 \frac{d\phi_i}{dx}\, k\, \frac{d\phi_j}{dx}\, dx.$$

- Same system! ("Natural" boundary conditions)
- A is $(n+1) \times (n+1)$

Inhomogeneous Neumann Condition (1/4)

Example:

$$-\frac{d^2\tilde{u}}{dx^2} = f(x) \quad \begin{cases} \tilde{u}(-1) = 0 \\ \tilde{u}'(1) = g \end{cases}$$

Standard Derivation: *Find $u \in X_0^N$ such that*

$$-\int_{-1}^{1} v\,\frac{d^2u}{dx^2}\,dx = \int_{-1}^{1} v\,f(x)\,dx \;\; \forall\, v \in X_0^N$$

Integrate by parts and use inhomogeneous Neumann condition:

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx - v\,\frac{du}{dx}\Big|_{-1}^{1} = \int_{-1}^{1} v\,f(x)\,dx$$

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx - \left[v(1)g - \cancel{v(-1)\frac{du}{dx}\Big|_{-1}} \right] = \int_{-1}^{1} v\,f(x)\,dx$$

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx = \int_{-1}^{1} v\,f(x)\,dx + v(1)g.$$

Inhomogeneous Neumann Condition (2/4)

- This equation is standard, save for the addition of the extra term at $x{=}1$:

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx \;=\; \int_{-1}^{1} v\,f(x)\,dx \;+\; v(1)g \;\; \forall v \in X_0^N.$$

- Consider now N equations, generated by taking for $i = 1, \ldots, N$,

$$v(x) = \phi_i(x) = h_i(x),$$

  where we are taking the basis functions to be the cardinal Lagrange polynomials, $h_i(x)$ for $x \in [-1,1] = \hat{\Omega}$.

- Note that all basis functions vanish at $x = 1$ except for $h_N(x)$, so only the **last equation** is modified.

Inhomogeneous Neumann Condition (3/4)

- Only the **last equation** is modified.

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx \;=\; \int_{-1}^{1} v\,f(x)\,dx \;+\; v(1)g \quad \forall v \in X_0^N.$$

- Accounting for the homogeneous Dirichlet condition at $x = -1$,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}; \;=\; \begin{pmatrix} \rho_1 f_1 \\ \rho_2 f_2 \\ \vdots \\ \rho_N f_N \;+\; g \end{pmatrix}.$$

- Though not immediately evident, we this system implies:

$$\lim_{N \longrightarrow \infty} \left.\frac{du}{dx}\right|_{x=1} \;=\; g$$

- To see this, we start with the top equation and integrate back.

Inhomogeneous Neumann Condition (4/4)

- Integrating the LHS of

$$\int_{-1}^{1} \frac{dv}{dx}\frac{du}{dx}\,dx \;=\; \int_{-1}^{1} v\,f(x)\,dx \;+\; v(1)g \;\;\forall v \in X_0^N.$$

  by parts and rearranging, we obtain

$$\int_{-1}^{1} v\left[-\frac{d^2u}{dx^2} - f\right] dx \;+\; v\,\frac{du}{dx}\bigg|_{x=1} \;=\; v(1)g \;\;\forall v \in X_0^N.$$

- Because our quadrature rule is exact for the 2nd-order term, the last equation of the preceding slide reads

$$\rho_N\left[-\frac{d^2u}{dx^2} - f\right] \;+\; v\,\frac{du}{dx}\bigg|_{x=1} \;=\; v(1)g.$$

- Since $\rho_N \sim 2N^{-2}$ we have: $\displaystyle\lim_{N\longrightarrow\infty} \frac{du}{dx}\bigg|_{x=1} \;=\; g.$

Inhomogeneous Dirichlet Condition

Example: $-\dfrac{d^2\tilde{u}}{dx^2} \;=\; f(x) \qquad \begin{cases} \tilde{u}(-1) & = \;\alpha \\ \tilde{u}'(1) & = \;g \end{cases}$

Standard Derivation: Rewrite as

$$-\frac{d^2 u_0}{dx^2} \;=\; f(x) \;+\; \frac{d^2 u_b}{dx^2}$$

and proceed in usual way, setting

$$u \;=\; u_0 \;+\; u_b$$

for any $u_b \in X_b^N$ satisfying $u_b(-1) = \alpha$.

Questions that arise:

- Implementation
- Cost (strongly dependent on number of space dimensions)
- Accuracy
- Spectrum
- Other (e.g., optimality)

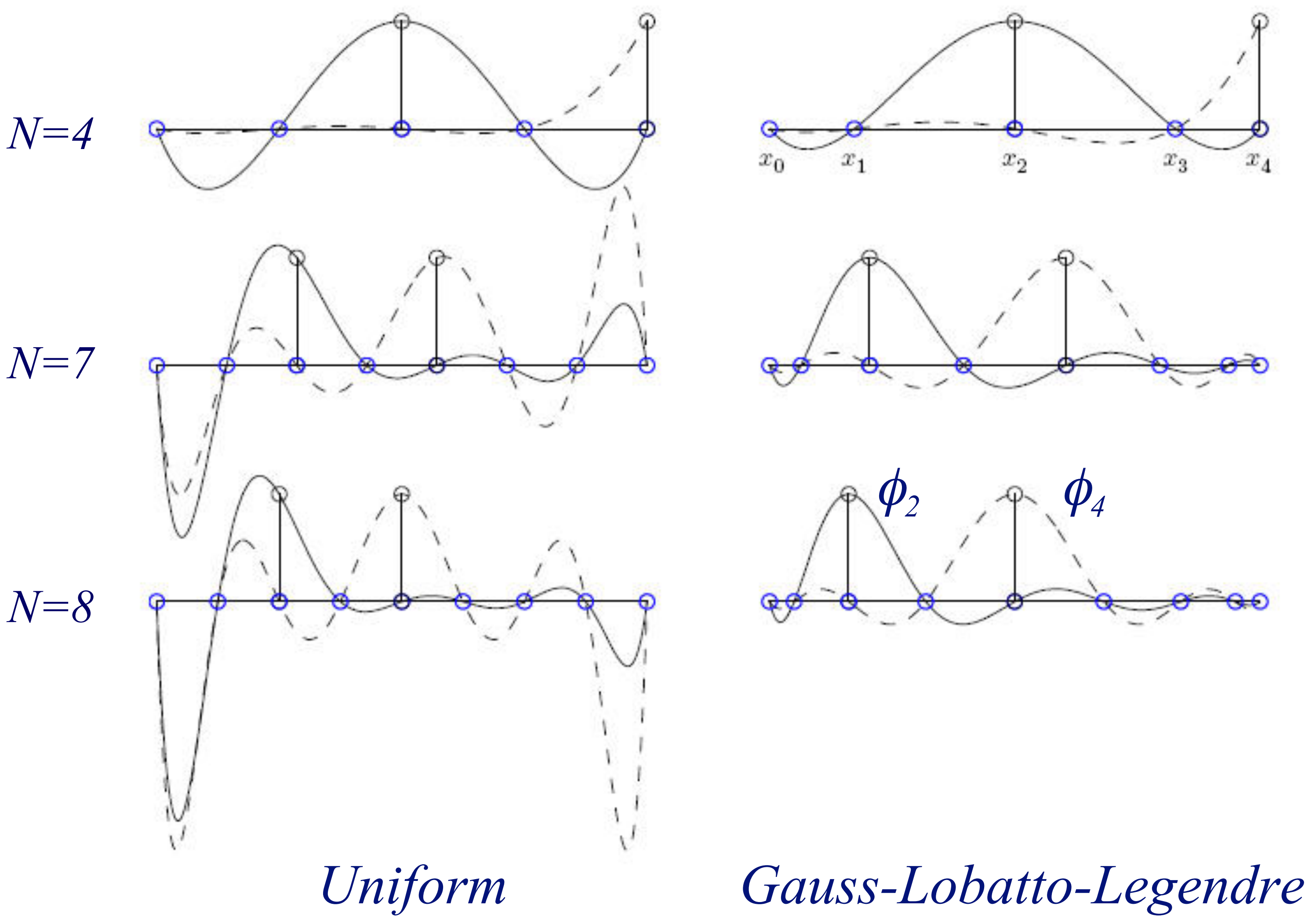
Choice of Spaces & Bases

- *An important choice is the **space**, X_0^N, and associated **basis** $\{ \phi_i \}$.*
- The former influences convergence, i.e.,
  - How large or small n must be for a given error.
- The latter influences implementation, i.e.,
  - details and level of complexity, and
  - performance (time to solution, for a given error)

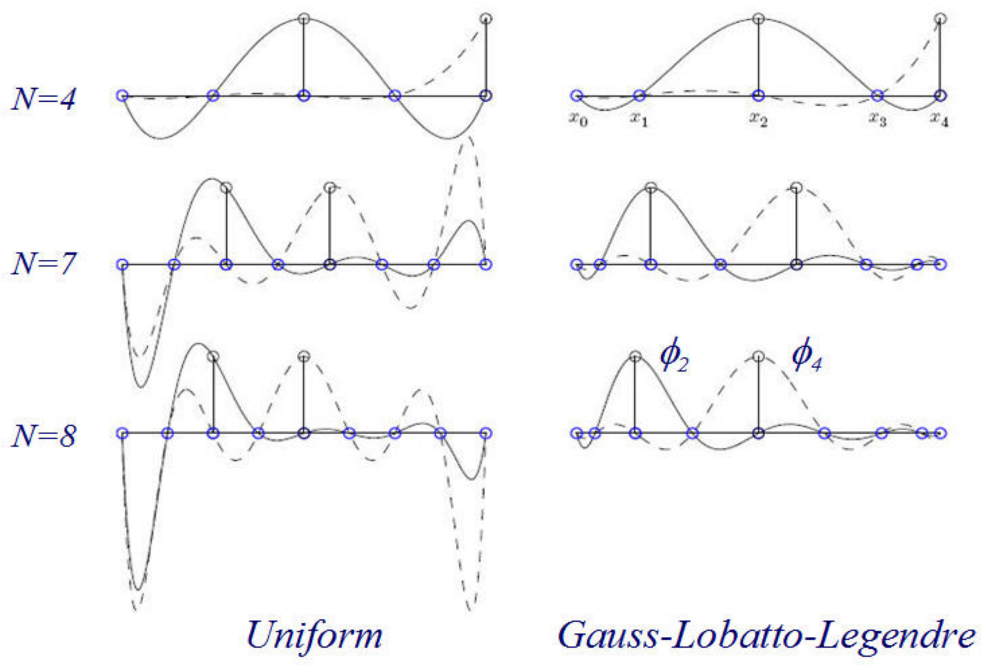
Unstable and Stable Bases within the Elements

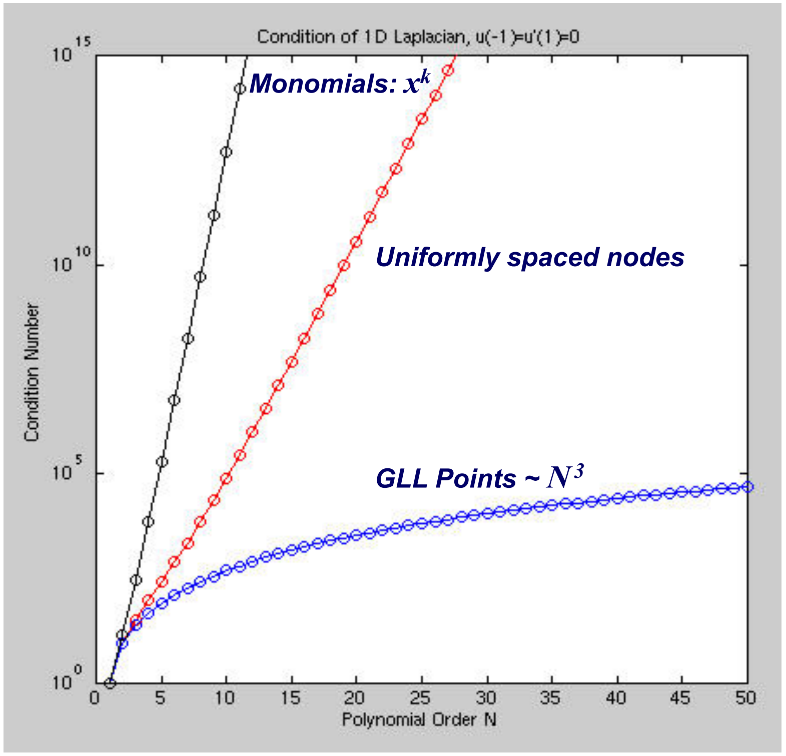
- *Examples of unstable bases are:*
  - Monomials (modal): $\phi_i = x^i$
  - High-order Lagrange interpolants (nodal) on *uniformly-spaced* points.
- Examples of *stable* bases are:
  - Orthogonal polynomials (modal), e.g.,
    - Legendre polynomials: $L_k(x)$, or
    - bubble functions: $\phi_k(x) := L_{k+1}(x) - L_{k-1}(x)$.
  - Lagrange (nodal) polynomials based on Gauss quadrature points (e.g., Gauss-Legendre, Gauss-Chebyshev, Gauss-Lobatto-Legendre, etc.)

Lagrange Polynomials: Good and Bad Point Distributions



Condition Number of A vs. Polynomial Order





- *Monomials and Lagrange interpolants on uniform points exhibit exponentional growth in condition number.*
- *With just a 7x7 system the monomials would lose 10 significant digits (of 15, in 64-bit arithmetic).*

Implementation, Start to Finish

- Assume $\Omega = [-1,1]$ and we have the two-point BVP

$$-\frac{d}{dx}\nu(x)\frac{du}{dx} \;=\; f, \quad u(-1) = u(1) = 0.$$

- WRT: *Find* $u \in X_0^N$ *such that*

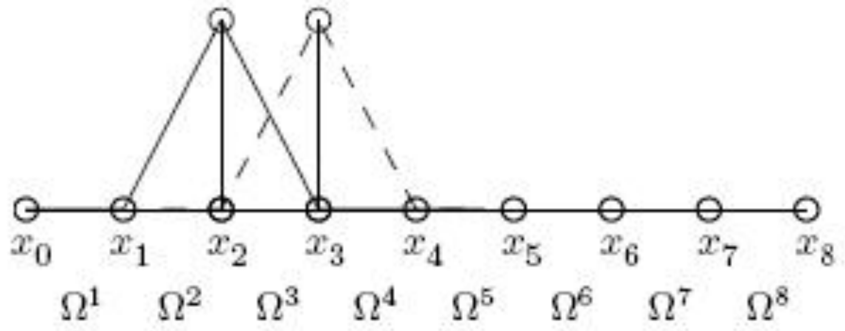
$$\begin{aligned}
a(v,u) &= (v,f) \quad \forall\, v \;\in\; X_0^N \\
a(\phi_i,u) &= (\phi_i,f) \quad i = 1,\ldots,N-1 \\
a(\phi_i,u) &= a\left(\phi_i, \sum_{j=1}^{N-1} u_j\phi_j\right) \;=\; \sum_{j=1}^{N-1} u_j a\left(\phi_i,\phi_j\right) \\
&= \sum_{j=1}^{N-1} a_{ij}u_j \\
a_{ij} &= a\left(\phi_i,\phi_j\right) \;=\; \int_{-1}^{1} \frac{d\phi_i}{dx}\nu(x)\frac{d\phi_j}{dx}\,dx
\end{aligned}$$

- Each a_{ij} is an integral:

$$
\begin{aligned}
a_{ij} &= \int_{-1}^{1} \frac{d\phi_i}{dx}\nu(x)\frac{d\phi_j}{dx}\,dx \\
&\approx \sum_{k=0}^{N} \rho_k \left.\frac{d\phi_i}{dx}\right|_{\xi_k} \nu(\xi_k) \left.\frac{d\phi_j}{dx}\right|_{\xi_k} \\
&= \sum_{k=0}^{N} \rho_k \hat{D}_{ki}\hat{B}_k\nu_k\hat{D}_{kj} \qquad\qquad (\hat{B} := \mathrm{diag}(\rho_k)\,) \\
A &= \hat{D}^T\left(\nu\hat{B}\right)\hat{D}.
\end{aligned}
$$

$\boxed{\textit{var\_coef.m}}$

Piecewise Polynomial Bases: Linear and Quadratic


x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8
Ω^1 Ω^2 Ω^3 Ω^4 Ω^5 Ω^6 Ω^7 Ω^8

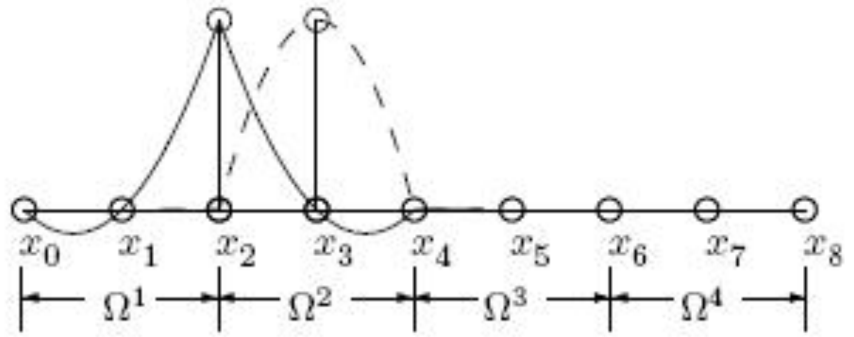
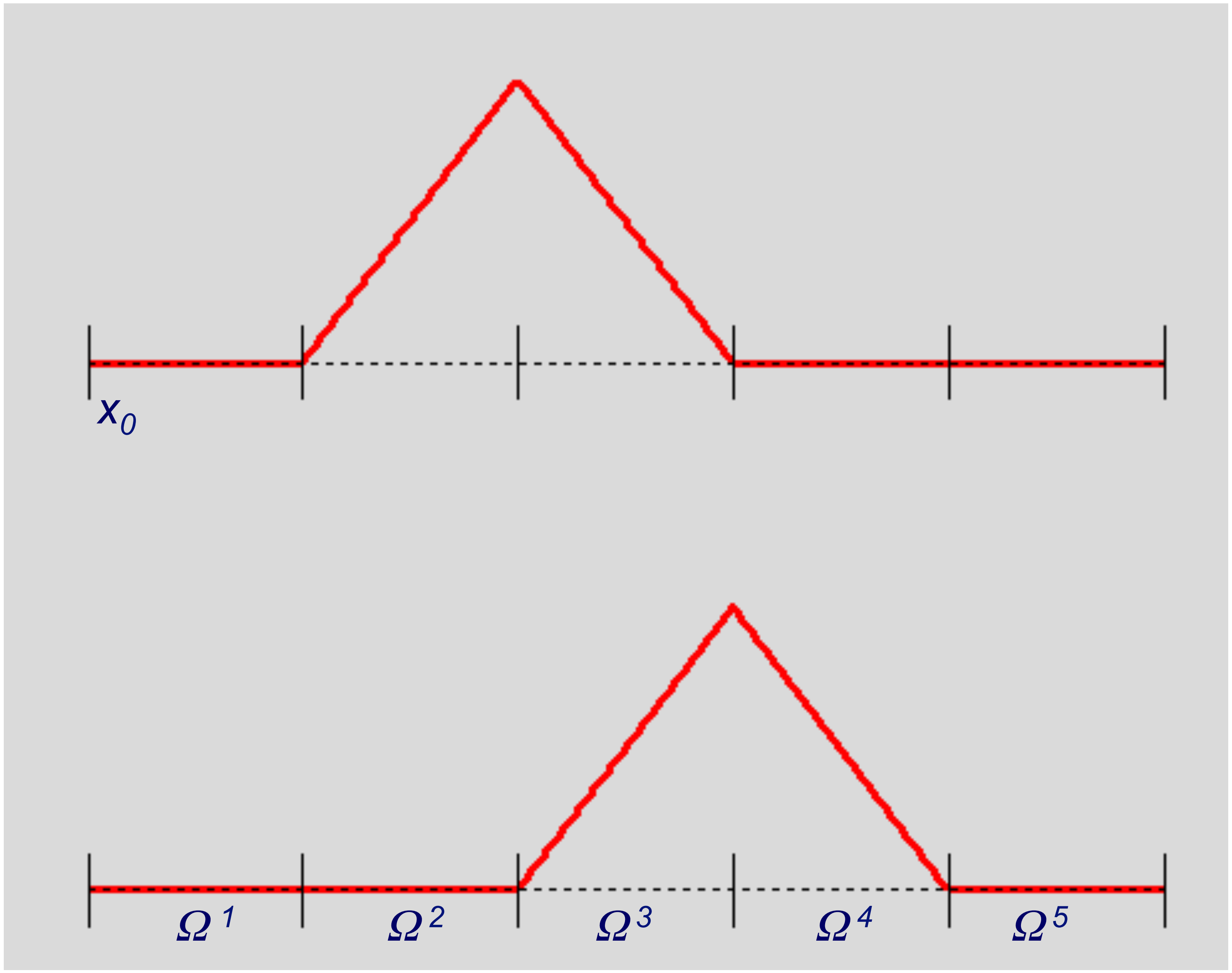

x_0 x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8
Ω^1 Ω^2 Ω^3 Ω^4

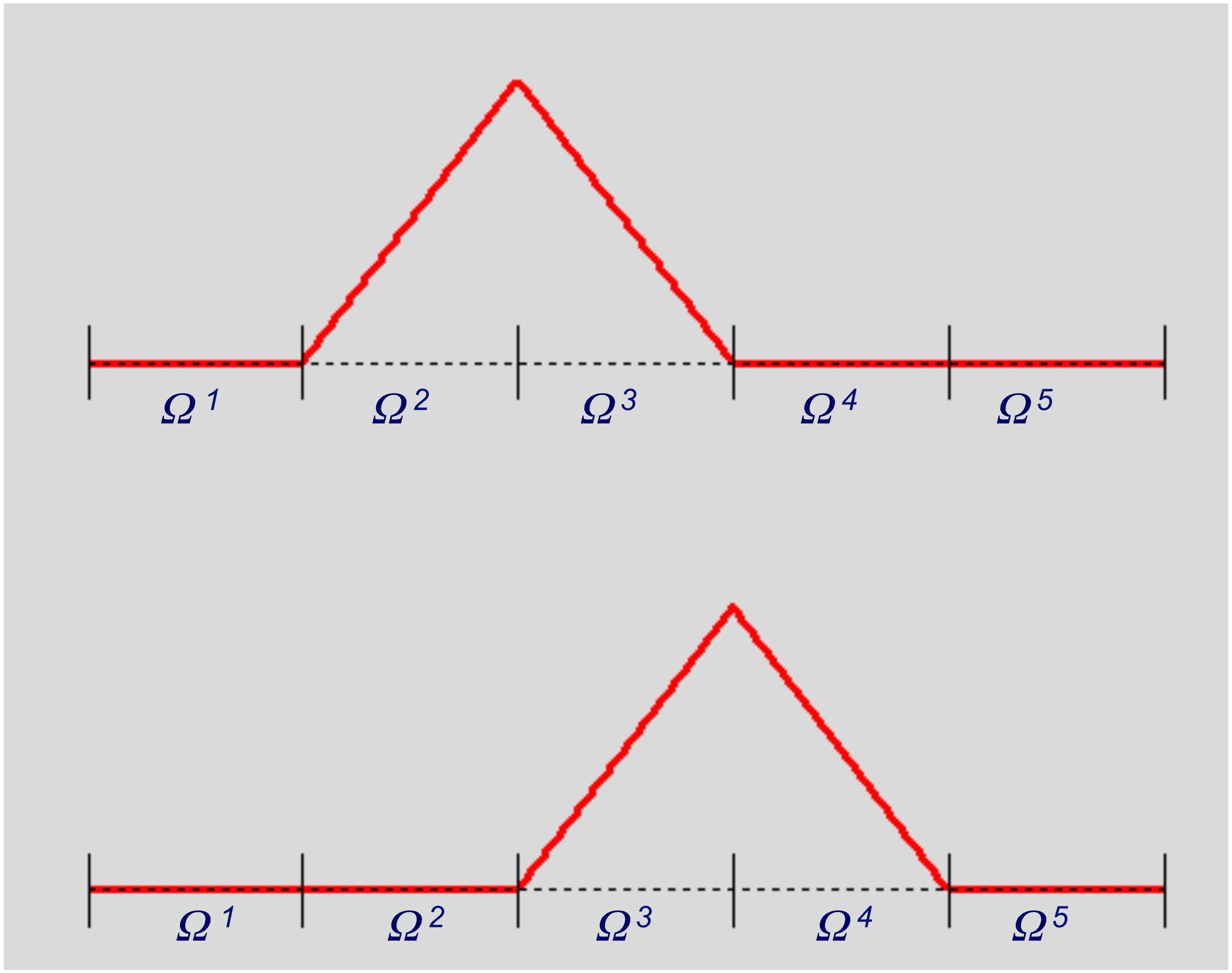
Figure 2: Examples of one-dimensional piecewise linear (left) and piecewise quadratic (right) Lagrangian basis functions, $\phi_2(x)$ and $\phi_3(x)$, with associated element support, Ω^e, $e = 1, \ldots, E$.

- Linear case results in A being tridiagonal (b.w. = 1)
- Q: What is matrix bandwidth for piecewise quadratic case?

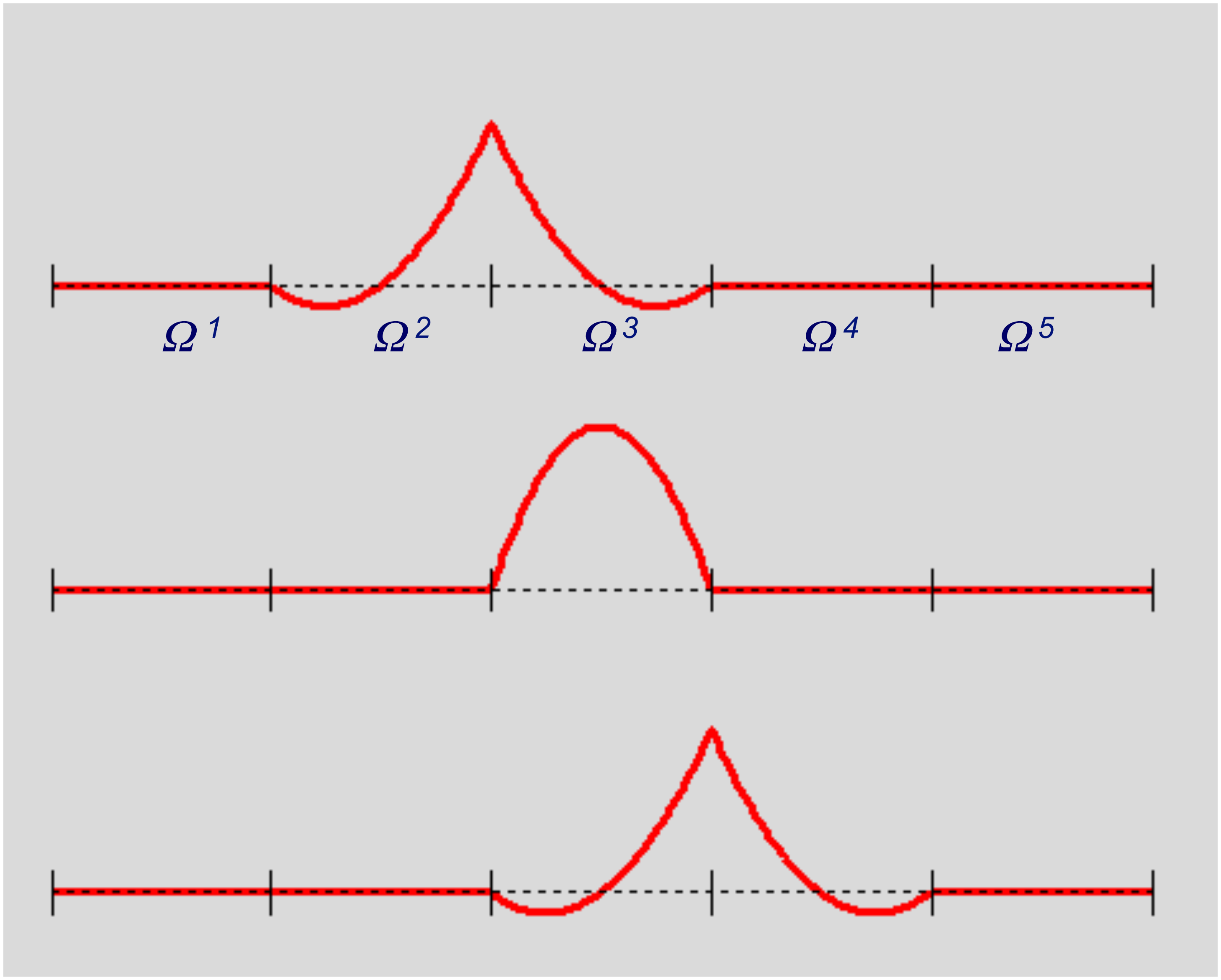
Basis functions for N=1, E=5 on element 3.



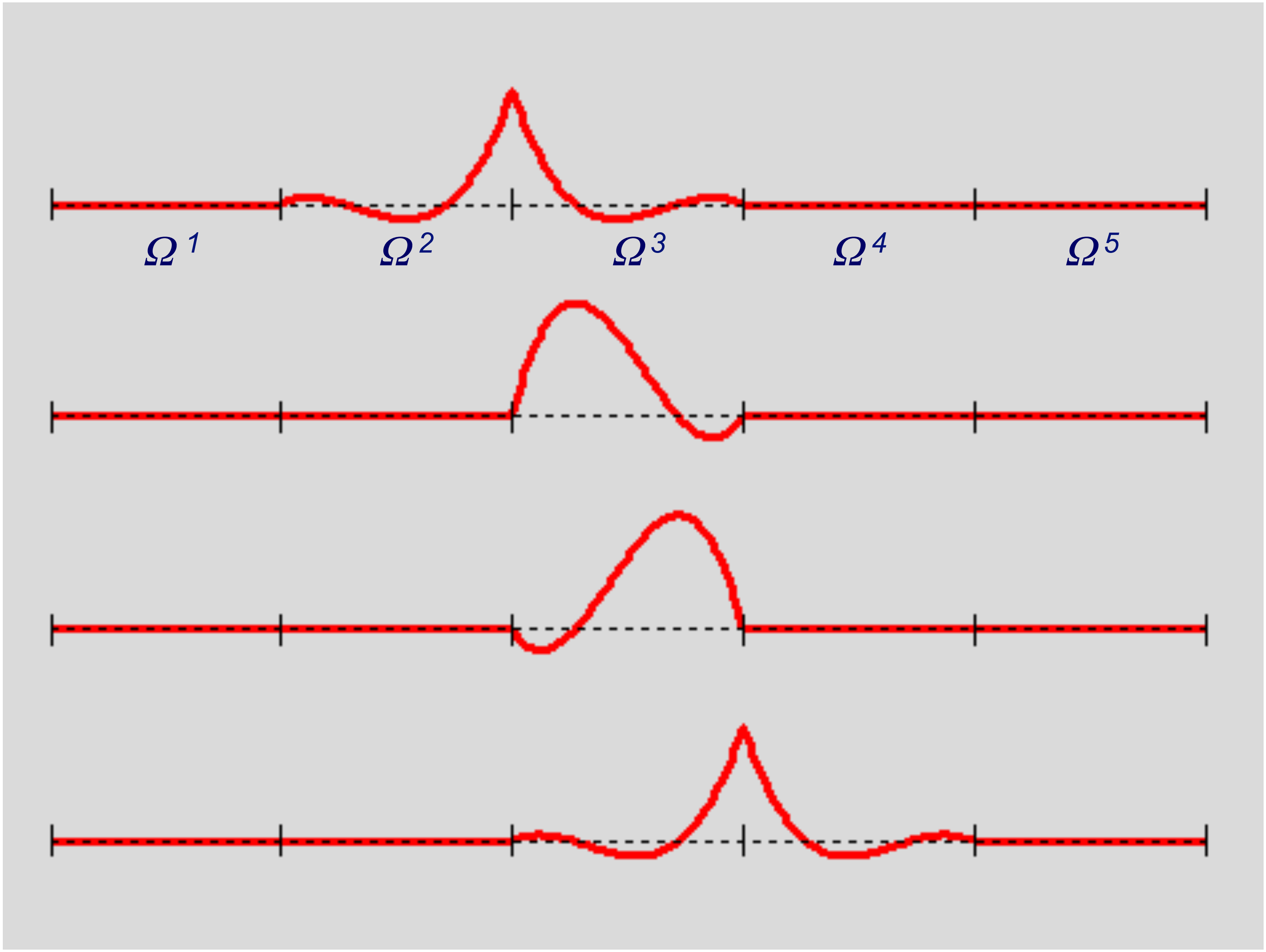
Basis functions for N=1, E=5 on element 3.



Basis functions for N=2, E=5



Basis functions for N=3, E=5



Important Properties of the Galerkin Formulation

- An essential property of the Galerkin formulation for the Poisson equation is that the solution is the ***best fit*** in the approximation space, with respect to the energy norm.

  Specifically, we consider the bilinear form,

$$a(v,u) \;:=\; \int_0^1 \frac{dv}{dx}\frac{du}{dx}\,dx,$$

  and associated semi-norm,

$$||u||_a^2 \;:=\; a(u,u),$$

  which is in fact a norm for all u satisfying the boundary conditions.

- It is straightforward to show that our Galerkin solution, u, is the closest solution to the exact $\tilde{u}$ in the a-norm. That is,

$$\boxed{\;||\, u - \tilde{u}\, ||_a \;\leq\; ||\, w - \tilde{u}\, ||_a \quad \textit{for all } w \in X_0^N\;}$$

- In fact, u is closer to $\tilde{u}$ than the interpolant of $\tilde{u}$.

Best Fit Property, 1/4

Define:

$$\begin{aligned}
\mathcal{L}^2_\Omega &= \left\{ v : \int_\Omega v^2 \, dx < \infty \right\} \\
\mathcal{H}^1 &= \left\{ v : \; v \in \mathcal{L}^2_\Omega, \; \int_\Omega (v')^2 \, dx < \infty \right\} \\
\mathcal{H}^1_0 &= \left\{ v : \; v \in \mathcal{H}^1, \; v|_{\partial\Omega} = 0 \right\}
\end{aligned}$$

Then, $\forall u, v \in \mathcal{H}^1_0$,

$$\begin{aligned}
a(u,v) &:= \int_\Omega u'v' \, dx && (a \text{ inner-product}) \\
||v||_a &:= \sqrt{a(v,v)} && (a\text{-norm}) \\
||\alpha v||_a &= |\alpha| \sqrt{a(v,v)} && \alpha \in \mathbb{R} \\
||v||_a &= 0 \text{ iff } v \equiv 0.
\end{aligned}$$

Best Fit Property, 2/4

We now demonstrate that $||u - \tilde{u}||_a \leq ||w - \tilde{u}||_a \; \forall w \in X_0^N$.

Let $e := u - \tilde{u}$ and $v := w - u \in X_0^N$.

For any $w \in X_0^N$ we have

$$\begin{aligned}
||w - \tilde{u}||_a^2 &= ||v + u - \tilde{u}||_a^2 \\
&= ||v + e||_a^2 \\
&= \int_0^1 (v+e)' \, (v+e)' \, dx \\
&= \int_0^1 (v')^2 dx \; + \; 2 \int_0^1 v' \, e' dx \; + \; \int_0^1 (e')^2 dx
\end{aligned}$$

Best Fit Property, 3/4

The second term vanishes:

$$\begin{aligned}
\int_0^1 v'\, e' dx + \quad &= \int_0^1 v'\,(u - \tilde{u})' dx \\
&= \int_0^1 v'\, u'\, dx \;-\; \int_0^1 v'\, \tilde{u}'\, dx \\
&= \int_0^1 v'\, u'\, dx \;+\; \int_0^1 v\, \tilde{u}''\, dx \;-\; v\, \tilde{u}'\big|_0^1 \\
&= \int_0^1 v'\, u'\, dx \;-\; \int_0^1 v\, f\, dx \\
&= 0 \qquad \forall\, v \in X_0^N,
\end{aligned}$$

by construction of u.

Best Fit Property 4/4

In summary, for any $w \in X_0^N$ we hawe

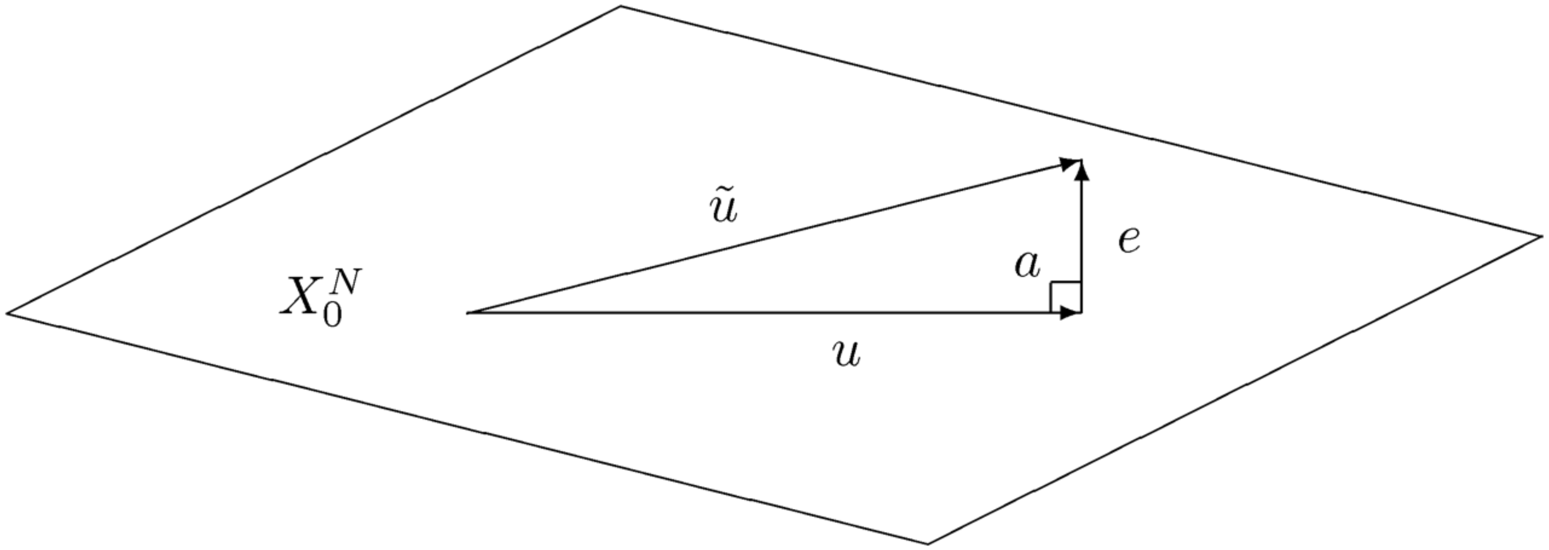
$$\begin{aligned} ||w - \tilde{u}||_a^2 &= ||v + u - \tilde{u}||_a^2 \\ &= ||v + e||_a^2 \\ &= \int_0^1 (v')^2 dx \; + \; 2 \int_0^1 \cancel{v' e' dx} \; + \; \int_0^1 (e')^2 dx \\ &= \int_0^1 (v')^2 dx \; + \; \int_0^1 (e')^2 dx \\ &\geq \int_0^1 (e')^2 dx \; = \; ||u - \tilde{u}||_a^2 \end{aligned}$$

Thus, of *all* functions in X_0^N, u is the *closest* to $\tilde{u}$ in the a-norm.

A deeper analysis establishes that, for $\tilde{u}$ analytic and $X^N = \mathbb{P}_N$, one has

$$||\tilde{u} - u||_{\mathcal{H}_0^1} \; \leq \; C \, e^{-\gamma N}$$

Best Fit Viewed as a Projection



- Note that this result also demonstrates that $a(v, e) = 0$ for all $v \in X_0^N$.
- That is, the Galerkin statement is equivalent to having the *error*,

$$e := u - \tilde{u} \perp_a X_0^N .$$

- Thus, u is the *projection* of $\tilde{u}$ onto X_0^N in the a inner-product.
- The procedure is often referred to as a Galerkin projection.

Best Fit Property Summary

- A significant advantage of the WRT over collocation is that the choice of *basis* for a **given** approximation space, X_0^N, is immaterial – you will get the same answer for any equivalent basis, modulo round-off considerations, because of the ***best fit property***.
- That is, the choice of ϕ_i influences the condition number of A, but would give the same answer (in inifinite-precision arithmetic) whether one used Lagrange polynomials on uniform points, Chebyshev points, or even monomial bases.
- This is not the case for collocation – so WRT is much more robust.
- Of course, Lagrange polynomials on Chebyshev or Gauss-Lobatto-Legndre points are preferred from a conditioning standpoint.

Eigenvalue Problems

- Standard eigenvalue problem for second-order ODE has form
$$u'' = \lambda f(t, u, u'), \qquad a < t < b$$
with BC
$$u(a) = \alpha, \qquad u(b) = \beta$$
where we seek not only solution u but also parameter λ
- Scalar λ (possibly complex) is *eigenvalue* and solution u is corresponding *eigenfunction* for this two-point BVP
- Discretization of eigenvalue problem for ODE results in algebraic eigenvalue problem whose solution approximates that of original problem

Example: Eigenvalue Problem

- Consider linear two-point BVP

$$u'' = \lambda g(t) u, \qquad a < t < b$$

with BC

$$u(a) = 0, \qquad u(b) = 0$$

- Introduce discrete mesh points t_i in interval $[a, b]$, with mesh spacing h and use standard finite difference approximation for second derivative to obtain algebraic system

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \lambda g_i y_i, \quad i = 1, \dots, n$$

where $y_i = u(t_i)$ and $g_i = g(t_i)$, and from BC $y_0 = u(a) = 0$ and $y_{n+1} = u(b) = 0$

Example, continued

- Assuming $g_i \neq 0$, divide equation i by g_i for $i = 1, \dots, n$, to obtain linear system

$$\boldsymbol{A}\boldsymbol{y} = \lambda \boldsymbol{y}$$

where $n \times n$ matrix $\boldsymbol{A}$ has tridiagonal form

$$\boldsymbol{A} = \frac{1}{h^2} \begin{bmatrix} -2/g_1 & 1/g_1 & 0 & \cdots & 0 \\ 1/g_2 & -2/g_2 & 1/g_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1/g_{n-1} & -2/g_{n-1} & 1/g_{n-1} \\ 0 & \cdots & 0 & 1/g_n & -2/g_n \end{bmatrix}$$

- This standard algebraic eigenvalue problem can be solved by methods discussed previously