Recap: Norms

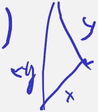
What's a norm?

$f: \mathbb{R}^n \to \mathbb{R}_0^+$ $\|\vec{x}\|$

Define *norm*.

A Function $\|\cdot\|: \mathbb{R}_n \to \mathbb{R}_n^+$

1. $\|\vec{x}\| > 0 \iff \vec{x} \neq \vec{0}$ $(\vec{x} \in \mathbb{R}_n)$
2. $\|\gamma x\| = |\gamma| \, \|x\|$ $(x \in \mathbb{R}_n, \gamma \in \mathbb{R})$
3. Δ-Ineq. $\|x+y\| \leq \|x\| + \|y\|$



Norms: Examples

Euclidean norm
$(p=2)$

Examples of norms?

p-norms

$$\left\|\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\|_p = \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}$$

$(p \geq 1)$ or $p = \infty$

Demo: Vector Norms [cleared]

Norms: Which one?

Does the choice of norm really matter much?

In finite dimensions, all vector norms are equiv.

$\|\cdot\|, \|\cdot\|^*$ norms $\Rightarrow \exists\, \alpha, \beta > 0$

$$\alpha \|x\| \leq \|x\|^* \leq \beta \|x\| \quad (x \in \mathbb{R}^n)$$

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

true value : $\vec{x}_0$
computed value : $\tilde{\vec{x}}$

abs. error: $\|\vec{x}_0 - \tilde{\vec{x}}\|$ ⇐

NO: $\cancel{\|\vec{x}_0\| - \|\tilde{\vec{x}}\|}$ WRONG! BAD!

Forward/Backward Error

Suppose *want* to compute $y = f(x)$, but *approximate* $\hat{y} = \hat{f}(x)$.

What are the forward error and the backward error?

$y = f(x)$

$$
\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "\hat{f}"] & f(x) \\
\hat{x} \arrow[r, "f"'] & \hat{f}(x) = \hat{y}
\end{tikzcd}
$$

backward error

forward error.

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the forward error?

$$\sqrt{2} \approx 1.41421$$

$$\|\Delta y\| = |1.4 - 1.41421\ldots| \approx 0.01421$$

$$\frac{|\Delta y|}{|y|} \approx 1\%$$

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the backward error?

$$\hat{x} = 1.4^2 = 1.96$$

$$\left|\frac{\Delta x}{x}\right| = \left|\frac{0.04}{2}\right| = 2\%$$

Forward/Backward Error: Observations

What do you observe about the relative manitude of the relative errors?

want the ratio of $\dfrac{\text{forward error}}{\text{backward error}}$ to be small.

Sensitivity and Conditioning

What can we say about amplification of error? k κ

$$\rightarrow \underbrace{|\text{rel. fwd error}|}^{\le 1} \le \kappa \cdot |\text{rel. backw. error}|^{\frac{1}{1000}}$$

$\uparrow$ rel. condition number "kappa"

$$\kappa = \max_x \frac{|\Delta y| / |y|}{|\Delta x| / |x|}$$
$\leftarrow$ fw.
$\leftarrow$ bw.

If cond nr. is small: "well-cond. problem"
large: "ill-conditioned"

$\underbrace{3.14}15\underline{9} \rightsquigarrow 3.14$

Example: Condition Number of Evaluating a Function

$$\kappa = \max_x \frac{|\Delta y| / |y|}{|\Delta x| / |x|}$$

$y = f(x)$. Assume f differentiable.

Forward error: $\Delta y = f(x+\Delta x) - f(x) \approx f'(x) \cdot \Delta x$

Demo: Conditioning of Evaluating tan [cleared]

$$\kappa \geq \frac{|\Delta y|/|y|}{|\Delta x|/|x|} \approx \frac{|f'(x)| \cdot \cancel{|\Delta x|} / |f(x)|}{\cancel{|\Delta x|} / |x|} = \frac{|f'(x) \cdot x|}{|f(x)|}$$

Stability and Accuracy

Previously: Considered *problems* or *questions*.

Next: Considered *methods*, i.e. computational approaches to find solutions.

When is a method *accurate*?

$$\| f(x) - \hat{f}(x) \| \text{ small}$$

When is a method *stable*?

- $\hat{x} = f^{-1}(\hat{f}(x))$ is close to x.
- stronger than "small condition number"