Simpler Problems: Triangular

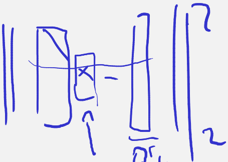
$\|Ax-b\|_2 \qquad A = QR$ → upper triangular, → orthogonal

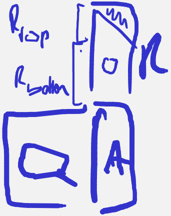
What do we win from transforming a least-squares system to upper triangular form?

$$\|Ax-b\|_2 = \|Q^T(Ax-b)\|_2 = \|Q^T(QRx-b)\|_2 = \|Rx-Q^Tb\|_2$$

→ min

How would we minimize the residual norm?





$$= \underbrace{\|(Q^Tb)_{top} - R_{top}x\|_2^2}_{0} + \underbrace{\|(Q^Tb)_{bottom}\|_2^2}$$

$(Q^Tb)_{top} \qquad (Q^Tb)_{bottom}$

Computing QR

- Gram-Schmidt
- Householder Reflectors
- Givens Rotations

Demo: Gram-Schmidt–The Movie [cleared]
Demo: Gram-Schmidt and Modified Gram-Schmidt [cleared]
Demo: Keeping track of coefficients in Gram-Schmidt [cleared]
Seen: Even modified Gram-Schmidt still unsatisfactory in finite precision arithmetic because of roundoff.

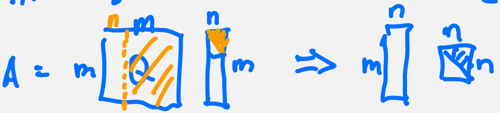
NOTE: Textbook makes further modification to ‘modified’ Gram-Schmidt:

- Orthogonalize *subsequent* rather than *preceding* vectors.
- Numerically: no difference, but sometimes algorithmically helpful.

Economical/Reduced QR

Is QR with square Q for $A \in \mathbb{R}^{m\times n}$ with $m > n$ efficient?

$$\|Ax-b\|_2 = \|Q(Rx - Q^T b)\|_2$$



In-Class Activity: QR

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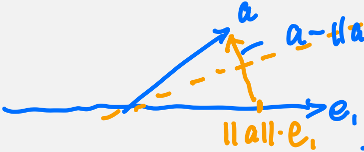
Householder Transformations

Find an *orthogonal* matrix Q to zero out the lower part of a vector $\boldsymbol{a}$.



$a - \|a\| \cdot e_1 = v$

$$I - 2\frac{v}{\|v\|}\frac{v^T}{\|v\|} = I - 2\frac{vv^T}{v^Tv} = H$$

Householder reflector.

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H\boldsymbol{a} = \pm \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1.$$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\boldsymbol{e}_i$ above.
- A product $H_n \cdots H_1 A = R$ of Householders makes it easy (and quite efficient!) to build a QR factorization.
- It turns out $\boldsymbol{v}' = \boldsymbol{a} + \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1$ works out, too–just pick whichever one causes less cancellation.
- H is symmetric
- H is orthogonal

Demo: 3x3 Householder demo [cleared]

Givens Rotations

If reflections work, can we make rotations work, too?

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sqrt{a_1^2 + a_2^2} \\ 0 \end{bmatrix}$$

(produces 1 zero at a time)

Demo: 3x3 Givens demo [cleared]

Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?