Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

$$A = Q\,R$$

R → small diagonal entries

"rank-revealing" / "column-pivoted" $\Big\}$ QR → HW14

Rank-Deficient Matrices and Least-Squares

What happens with Least Squares for rank-deficient matrices?

$$A\boldsymbol{x} \cong \boldsymbol{b}$$

$$\|Ax-b\|_2 \to \min$$

$$n \in N(A)$$

$$\|A(x+n)-b\|_2 = \|Ax+b\|_2$$

$$\to \min \|x\|_2$$

"Total least squares"

SVD $A = U\Sigma V^T$

SVD: What's this thing good for? (I)

$$A = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_k & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$$

≥ 0 sorted $|\sigma_1| \geq |\sigma_2| \dots$

$$\|A\|_2 = \sigma_1$$

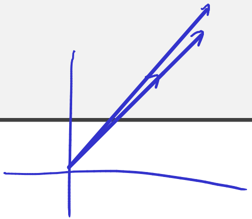
$$\operatorname{cond}_2(A) = \sigma_1/\sigma_n$$

$$N(A) = \operatorname{span}(v_{k+1}, \dots, v_n)$$

$$N(\Sigma) = \operatorname{span}\left((0 \cdots 0 \underset{\uparrow_{k+1}}{1} 0 0 0)^T, \dots, (0 \cdots 0 1)^T\right)$$

$$\operatorname{rank}(A) = k$$

$$\text{num-rank}(A, \varepsilon) = \#\{\sigma_i \geq \varepsilon\}$$



SVD: What's this thing good for? (II)

- *Low-rank Approximation*

Theorem (Eckart-Young-Mirsky)

If $k < r = \operatorname{rank}(A)$ *and*

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T, \quad \textit{then}$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k}^{n} \sigma_j^2}.$$

Demo: Image compression [cleared]

SVD: What's this thing good for? (III)

- The minimum norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$:

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

$$Ax \cong b \Leftrightarrow \Sigma V^T x \cong U^T b \qquad y = V^T x$$

$$\Sigma y \cong \underbrace{U^T b}_{c}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix} \qquad \Sigma^+ = \left[\begin{array}{ccc|ccc} \sigma_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \sigma_r^{-1} & & & \\ \hline & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \ddots \end{array}\right]$$

$$\|\Sigma y - c\|_2^2 = \sum_{i=1}^{r} (\sigma_i y_i - c_i)^2 + \underline{\underline{\sum_{i=r+1}^{n} c_i^2}}$$

$$\|y\|_2^2 = \underline{\sum_{i=1}^{r} y_i^2} + \sum_{i=r+1}^{n} y_i^2$$

SVD: Minimum-Norm, Pseudoinverse

What is the minimum 2-norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$ and why?

$y = \Sigma^{+} U^{T} b$ is the minimum norm solu. to $\Sigma y \cong U^{T} b$.

$x = Vy = \underbrace{V\Sigma^{+}U^{T}}_{\text{pseudoinverse.}} b$

Generalize the pseudoinverse to the case of a rank-deficient matrix.

when A full rank, $V\Sigma^{+}U^{T} = (A^{T}A)^{-1}A$

Comparing the Methods

Methods to solve least squares with A an $m \times n$ matrix:

① normal eqn.

A^TA, solve. $n^2m/2 + n^3/6$

② QR with Householder.

$mn^2 - n^3/3$

③ SVD: $mn^2 + n^3$

$O(mn^2 + n^3)$

Demo: Relative cost of matrix factorizations [cleared]

In-Class Activity: Householder, Givens, SVD

In-class activity: Householder, Givens, SVD

Outline

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq 0$ is called an *eigenvector* of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda \boldsymbol{x}.$$

- In that case, λ is called an *eigenvalue*.
- The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- The *spectral radius* is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max \{|\lambda| : \lambda(A)\}$$

Finding Eigenvalues

How do you find eigenvalues?

$$A\boldsymbol{x} = \lambda\boldsymbol{x} \Leftrightarrow (A - \lambda I)\boldsymbol{x} = 0$$
$$\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \geqslant 5$ is no general formula for roots of polynomial. IOW: no.

- For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *approximate*.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

Multiplicity

$\det|A - \lambda I| = (\lambda - 5)^2 (\lambda - 3)^4$

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- *Algebraic Multiplicity*: multiplicity of the root of the characteristic polynomial
- *Geometric Multiplicity*: #of lin. indep. eigenvectors

In general: AM $\geqslant$ GM.
If AM $>$ GM, the matrix is called *defective*.

An Example

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{bmatrix} 1+\varepsilon & 1 \\ & 1 \end{bmatrix}.$$

$|\lambda I - A| = (\lambda-1)^2$

$AM = 2$

$GM = 1 < AM.$

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 1\cdot\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x+y = x \\ y = y \end{matrix} \Rightarrow \begin{matrix} y = 0 \\ y = y \end{matrix}$$

$\begin{bmatrix} x \\ 0 \end{bmatrix}$

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