Diagonalizability

When is a matrix called *diagonalizable*?

$\det(A-\lambda I)$, if $\underline{AM} = GM$ then $(x_i)_{i=1}^n$

$X = [x_1, x_2, \cdots x_n]$

$AX = XD \Rightarrow \underline{A = XDX^{-1}}$

$\hookrightarrow \operatorname{diag}(\lambda_1, \cdots \lambda_n)$.

$\Rightarrow A$ is diagonalizable.

Similar Matrices

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

if $Av = \lambda v$, $B = X^{-1}AX$, then $w = X^{-1}v$ s.t.

$$Bw = (X^{-1}AX)(X^{-1}v) = X^{-1}\underline{Av} = \lambda w$$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $\boxed{A\boldsymbol{x} = \lambda \boldsymbol{x}}$ do?

Shift. $A \to A - \sigma I$

$$(A - \sigma I)x = (\lambda - \sigma)x$$

Inversion. $A \to A^{-1}$

$$A^{-1}(Ax) = A^{-1}(\lambda x) \Rightarrow A^{-1}x = \frac{1}{\lambda}x$$

Power. $A \to A^k$

$$A^k x = A^{k-1}Ax = \dots = \lambda^k x$$

Eigenvalue Transformations (II)

Polynomial $A \to aA^2 + bA + cI$

$$(aA^2 + bA + cI)x = (a\lambda^2 + b\lambda + c)x$$

Similarity $T^{-1}AT$ with T invertible

$$(T^{-1}AT)y = T^{-1}ATT^{-1}x = \lambda \underbrace{T^{-1}x}_{y}$$

$$\underline{y = T^{-1}x}$$

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \to A + E$.

diagonal?

What happens to the eigenvalues?

$$X^{-1}(A+E)X = D+F$$

not necessarily diagonal

- $A+E$, $D+F$ are similar $\Rightarrow$ same eigenvalues
- Suppose (μ, v) is an eigenpair of $D+F$ (or $A+E$)

$$(D+F)v = \mu v$$

$$Fv = (\mu I - D)v$$

$(\mu I - D)$ invertible because?

WLOG: μ not an eigenvalue of $A+E$

$$(\mu I - D)^{-1} F v = v$$

$$\Rightarrow \|\cancel{v}\| \leq \|(\mu I - D)^{-1}\| \, \|F\| \, \|\cancel{v}\|$$

$$\|(\mu I - D)^{-1}\|^{-1} \leq \|F\|$$

Sensitivity (II)

$X^{-1}(A+E)X = D+F$. Have $\left\|(\mu I - D)^{-1}\right\|^{-1} \leqslant \|F\|$.

$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\|(\mu I - D)^{-1}\|^{-1} = |\mu - \lambda_k|$ where λ_k is closest λ to μ.

$|\mu - \lambda_k| = \|(\mu I - D)^{-1}\|^{-1} \leqslant \|F\| = \|X^{-1} E X\|$
$\leqslant \|X\| \|X^{-1}\| \|E\|$
$= \operatorname{cond}(X) \|E\|$

- "bad" if X is ill-cond, i.e. if eigenvectors of A are nearly lin. dep.
- orthogonal eigenvectors yield perfect conditioning
  - ↳ foreshadow: symm eigval problems easier than nonsym.
- bound holds for all eigenvalues, may overestimate

Demo: Bauer-Fike Eigenvalue Sensitivity Bound [cleared]

Power Iteration

What are the eigenvalues of A^{1000}?

$Ax = \lambda x$ $\qquad y = 15x$

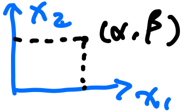
Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Further assume $\|\boldsymbol{x}_i\| = 1$.

$x = \alpha x_1 + \beta x_2$ $\qquad Ax = A(\alpha x_1 + \beta x_2) = \alpha \lambda_1 x_1 + \beta \lambda_2 x_2$

$\underbrace{A^{1000} x}_{y} = \alpha(\lambda_1^{1000} x_1) + \beta(\lambda_2^{1000} x_2)$ $\qquad | : \lambda_1^{1000}$

$\frac{y}{\lambda_1^{1000}} = \alpha_1 x_1 + \beta \underbrace{\underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)}_{<1}^{1000}}_{\ll 1} x_2$

Power Iteration: Issues?



What could go wrong with Power Iteration?

- $\|A^{1000}x\| \sim \alpha|\lambda_1|^{1000}$. fix: normalize at each iter.
- $\alpha = 0$, not an issue in practice.
- $|\lambda_1| > |\lambda_2|$ not true.

What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$Ax = \lambda_1 x$, $\lambda_1 \overset{?}{=} \frac{Ax}{x}$,

$\lambda_1 = \frac{x^T A x}{x^T x}$ Rayleigh quotient.

also an estimate of a "nearby" eigenvalue.

Convergence of Power Iteration

What can you say about the convergence of the power method?

Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$

$$e_{k+1} \sim \frac{|\lambda_2|}{|\lambda_1|} e_k \sim \left(\frac{|\lambda_2|}{|\lambda_1|}\right)^k e_1$$

$$A - \sigma I \quad \rightarrow \quad e_{k+1} \sim \frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} e_k .$$

Inverse Iteration

Describe *inverse iteration*.

$$(A - \sigma I)^{-1}$$

$$e_{k+1} \sim \frac{|\lambda_{closest} - \sigma|}{|\lambda_{2nd\text{-}colsest} - \sigma|} e_k$$

Rayleigh Quotient Iteration

Describe *Rayleigh Quotient Iteration*.

$$\sigma_k = \frac{x_k^T A x_k}{x_k^T x_k}$$

$$x_{k+1} = (A - \sigma_k I)^{-1} x_k.$$

Demo: Power Iteration and its Variants [cleared]