Truncation Error in Interpolation

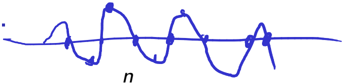
If f is n times continuously differentiable on a closed interval I and $p_{n-1}(x)$ is a polynomial of degree at most n that interpolates f at n distinct points $\{x_i\}$ $(i = 1, ..., n)$ in that interval, then for each x in the interval there exists ξ in that interval such that

$$\text{Error}\quad \underbrace{f(x) - p_{n-1}(x)}_{R(x)} = \frac{f^{(n)}(\xi)}{n!} \underbrace{(x - x_1)(x - x_2) \cdots (x - x_n)}_{W(x)}.$$

$$R(x) = f(x) - p_{n-1}(x)$$

$$Y_x(t) = R(t) - \frac{R(x)}{W(x)} W(t) \qquad (x \notin \{x_1, \ldots, x_n\})$$

Truncation Error in Interpolation: cont'd.



$$Y_x(t) = R(t) - \frac{R(x)}{W(x)} W(t) \quad \text{where} \quad W(t) = \prod_{i=1}^{n} (t - x_i)$$

- $Y_x(t)$ has $n+1$ roots. | $\{x_1, \dots, x_n\}$ because $R(x_i) = 0$, $W(x_i) = 0$

  $Y_x(x) = 0$

- Rolle's theorem says Y_x' has at least n roots (btw. the x_i)

  $Y_x^{(n)}$ has at least one root: ξ

  $$Y^{(n)}(t) = f^{(n)}(t) - \frac{R(x)}{W(x)} n!$$

  $$f^{(n)}(\xi) \cdot \frac{W(x)}{n!} = R(x)$$

  $R(t) = f - p_{n-1}$

Error Result: Connection to Chebyshev

What is the connection between the error result and Chebyshev interpolation?

$$f(x) - p_{n-1}(x) = \frac{f^{(n)}(\xi)}{n!} \prod_{i=1}^{n} (x - x_i)$$

$$\max_x |f(x) - p_{n-1}(x)| \le \underbrace{\max_x \left| \frac{f^{(n)}(x)}{n!} \right|}_{} \cdot \underline{\max_x \left| \prod_{i=1}^{n} (x - x_i) \right|}$$

$$\min_{\{x_i\}} \max_x \left| \prod_{i=1}^{n} (x - x_i) \right| \Rightarrow \text{Chebyshev nodes!}$$

$\neq$ "chebyshev approximating polynomial" p^*

$$\|f(x) - p_{n-1}(x)\| \le (1 + \Lambda_{n-1}) \|f(x) - p^*\|$$

Demo: Chebyshev Interpolation [cleared] (Part V)

Error Result: Simplified From

Boil the error result down to a simpler form.

$$\begin{cases} |f^{(n)}(x)| \le M, & x \in [x_1, x_n] \\ h = x_n - x_1 \end{cases}$$

$$E(h) := \max_x |f(x) - p_{n-1}(x)| \le C \cdot M \cdot h^n$$

$$\Rightarrow E(h) = O(h^n), \quad h \to 0$$

n-th order convergence.

Demo: Interpolation Error [cleared] **Demo:** Jump with Chebyshev Nodes [cleared]

Going piecewise: Simplest Case

Construct a piecweise linear interpolant at four points.

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
|	$f_1 = a_1 x + b_1$	|	$f_2 = a_2 x + b_2$	|	$f_3 = a_3 x + b_3$	|
|	2 unk.	|	2 unk.	|	2 unk.	|
|	$f_1(x_0) = y_0$	|	$f_2(x_1) = y_1$	|	$f_3(x_2) = y_2$	|
|	$f_1(x_1) = y_1$	|	$f_2(x_2) = y_2$	|	$f_3(x_3) = y_3$	|
|	2 eqn.	|	2 eqn.	|	2 eqn.	|

Why three intervals?

- 2 at end
- 1 in the middle (add more as needed)

Piecewise Cubic ('Splines')

x_0, y_0 | f_1: $a_1x^3 + b_1x^2 + c_1x + d_1$ | x_1, y_1 | f_2: $a_2x^3 + b_2x^2 + c_2x + d_2$ | x_2, y_2 | f_3: $a_3x^3 + b_3x^2 + c_3x + d_3$ | x_3, y_3

2·6

4 unk | 4 unk | 4 unk

$f_1(x_0) = y_0$

$f_1(x_1) = y_1$

$f_2(x_1) = y_1$

$f_2(x_2) = y_2$

$f_1'(x_1) = f_2'(x_1)$

$f_1''(x_1) = f_2''(x_1)$

$\left.\begin{array}{l} f_3(x_2) = y_2 \\ f_3(x_3) = y_3 \end{array}\right] 6$

$\left.\begin{array}{l} f_2'(x_2) = f_3'(x_2) \\ f_2''(x_2) = f_3''(x_2) \end{array}\right] 4$

$f_1''(x_0) = 0$

$f_3''(x_3) = 0$

"natural spline"

Piecewise Cubic ('Splines'): Accounting

x_0, y_0		x_1, y_1		x_2, y_2		x_3, y_3
	f_1		f_2		f_3	
	$a_1 x^3 + b_1 x^2 + c_1 x + d_1$		$a_2 x^3 + b_2 x^2 + c_2 x + d_2$		$a_3 x^3 + b_3 x^2 + c_3 x + d_3$	

Outline

Numerical Integration: About the Problem

What is numerical integration? (Or quadrature?)

$$I(f) = \int_a^b f(x)\,dx$$

What about existence and uniqueness?

- Riemann / Lebesgue.
- "f is piecewise continuous & bounded"

Conditioning

Derive the (absolute) condition number for numerical integration.