Properties of ODEs

$y^{(k)} = f(t, \underbrace{y, \ldots, y^{(k-1)}})$ $f(t, \vec{x})$

$f(t, y, \ldots, y^{(k)}) = 0$

What is a linear ODE?

$$f(t, \vec{x}) = A(t)\vec{x} + \vec{b}(t)$$

What is a linear and homogeneous ODE?

$$f(t, \vec{x}) = A(t)\vec{x}$$

What is a constant-coefficient ODE?

$$f(t, x) = A\vec{x} + \vec{b}$$

Properties of ODEs (II)

What is an autonomous ODE?

$f(t, \vec{x}) = f(\vec{x})$

$y_0'(t) = 1$, $y_0(0) = 0$

Existence and Uniqueness

$y' = \lambda \cdot y$

Consider the perturbed problem

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases} \qquad \begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

Then if $\boldsymbol{f}$ is *Lipschitz continuous* (has ‘bounded slope’), i.e.

$$\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{y})\| \leq L\,\|\boldsymbol{y} - \boldsymbol{y}\|$$

(where L is called the *Lipschitz constant*), then...

there exists a solution y in a neighborhood of t_0

$\|y(t) - \hat{y}(t)\| \leq e^{L(t-t_0)} \|y - \hat{y}_0\|$

What does this mean for uniqueness?

Implied by the bound. $(\|y - \hat{y}_0\| = 0)$

Conditioning

Unfortunate terminology accident: "Stability" in ODE-speak
To adapt to conventional terminology, we will use 'Stability' for

- the conditioning of the IVP, *and*
- the stability of the methods we cook up.

Some terminology:

An ODE is stable if and only if. . .

The solution is continuously dependent on initial data.

$\Leftrightarrow$ For all $\varepsilon > 0$ there exists a $\delta > 0$

$$\|\hat{y}_0 - y_0\| < \delta \Rightarrow \|\hat{y}(t) - y(t)\| < \varepsilon \text{ for all } t \geq t_0.$$

An ODE is asymptotically stable if and only if

$$\|\hat{y}(t) - y(t)\| \to 0 \quad (t \to \infty)$$

Example I: Scalar, Constant-Coefficient

$$\frac{y'}{y} = \lambda$$
$$\| $$
$$(\ln y)'$$

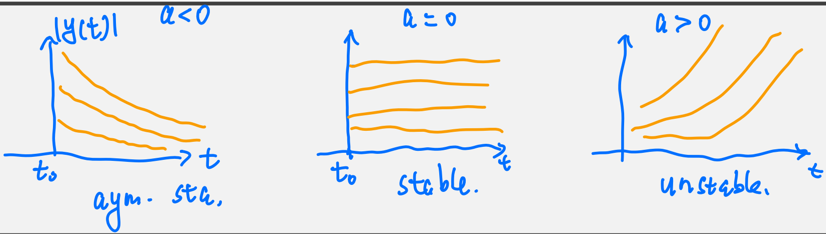
$$\begin{cases} y'(t) = \lambda y \\ y(0) = y_0 \end{cases} \quad \text{where} \quad \lambda = a + ib$$

Solution?

$$y(t) = y_0 \cdot e^{\lambda t} = y_0 e^{at} \underbrace{e^{ibt}}_{\|}$$

$$|\cos(bt) + i\sin(bt)| = 1$$

When is this stable? $a \leq 0$



Example II: Constant-Coefficient System

$A = VDV^{-1}$

$\Uparrow$

$$\begin{cases} \boldsymbol{y}'(t) = \underline{A}\boldsymbol{y}(t) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

$\Rightarrow y'(t) = VDV^{-1}y(t)$

$[\underbrace{V^{-1}y(t)}]' = D\,\underbrace{V^{-1}y(t)}_{w}$

Assume $V^{-1}AV = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ diagonal. Find a solution.

$$\begin{cases} w'(t) = Dw(t) \\ w(t_0) = V^{-1}y_0 \end{cases}$$

When is this stable?

when $\operatorname{Re}(\lambda_i) \leq 0$

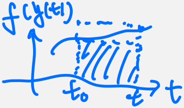
Euler's Method

Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

- Discrete times: $t_1, t_2, \ldots$, with $t_{i+1} = t_i + h$
- Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau))\,d\tau$$



Rectangle rule

$\Rightarrow$ Euler's method.

Euler's method: Forward and Backward

$y' = f(y)$

$\frac{y_{k+1} - y_k}{h} = f(y_k)$

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(\tau))\mathrm{d}\tau,$$

Volterra IE of the First kind

Use 'left rectangle rule' on integral:

$y(t_k) = y_k$

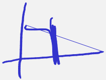
Fw Euler "explicit"

$y_{k+1} = y_k + h f(y_k)$ ⇐ need to eval expr.

↳ $h = t_{k+1} - t_k$ (assume constant)

Use 'right rectangle rule' on integral:

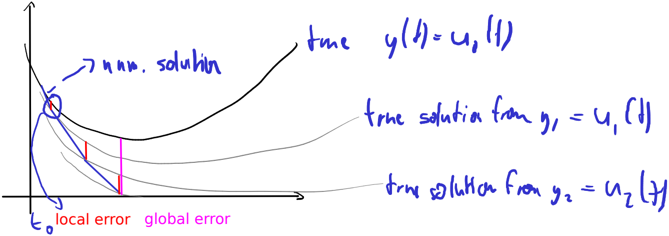
Bw Euler "implicit"



$y_{k+1} = y_k + h f(y_{k+1})$ ⇐ need to solve nonlin system

Demo: Forward Euler stability [cleared]

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$. Define the local error at step k as...

$$\boxed{l_k = y_k - u_{k-1}(t_k)}$$

Define the global error at step k as...

$$\boxed{g_k = y(t_k) - y_k}$$

About Local and Global Error

Is global error = local errors?

No.

cf. "compound interest"

global error not accounted for by $\sum$ local

"propagated error"

A time integrator is said to be *accurate of order p* if...

$$\ell_k = O(h^{p+1})$$

ODE IVP Solvers: Order of Accuracy

A time integrator is said to be *accurate of order p* if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect–why?

integrate to $t=1$, #steps $\frac{1}{h}$

$$\frac{1}{h} \cdot O(h^{p+1}) = O(h^{p})$$

$e^{rt} \approx 1 + rt$

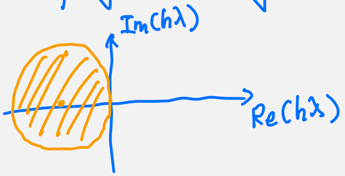
Stability of a Method

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_k = y_{k-1} + h \cdot \lambda y_{k-1}$$
$$= (1+h\lambda)^k y_0$$

$|1+h\lambda| \le 1 \Rightarrow$ stable. $\|y_k\| \le |1+h\lambda|^k \cdot \|y_0\|$

↳ amplification factor



Stability: Systems

What about stability for systems, i.e.

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t)?$$

$w = V^{-1} y$

$\Rightarrow |1 + h\lambda_i| \leq 1$ implies stability