Does a solution even exist? How sensitive are they?

General case is harder than root finding, and we couldn't say much there.
$\rightarrow$ Only consider linear BVP

$$(*)\begin{cases} \boldsymbol{y}'(x) = A(x)\boldsymbol{y}(x) + \boldsymbol{b}(x) \\ B_a\boldsymbol{y}(a) + B_b\boldsymbol{y}(b) = \boldsymbol{c} \end{cases}$$

$y'' = f \ldots$

$y(a) =$

$y(b) = 0$

To solve that, consider *homogeneous IVP*

$$\boldsymbol{y}_i'(x) = A(x)\boldsymbol{y}_i(x)$$

with initial condition

$$\boldsymbol{y}_i(a) = \boldsymbol{e}_i.$$

Note: $\boldsymbol{y} \neq \boldsymbol{y_i}$. $\boldsymbol{e}_i$ is the ith unit vector. With that, build the fundamental solution matrix

$$Y(x) = \begin{bmatrix} | & & | \\ \boldsymbol{y}_1 & \cdots & \boldsymbol{y}_n \\ | & & | \end{bmatrix}$$

ODE Systems: Existence

Let

$$Q := B_a Y(a) + B_b Y(b)$$

Then $(*)$ has a unique solution if and only if Q is invertible. Solve to find coefficients:

$$Q\boldsymbol{\alpha} = \boldsymbol{c}$$

Then $Y(x)\boldsymbol{\alpha}$ solves $(*)$ with $\boldsymbol{b}(x) = 0$.

Define $\Phi(x) := Y(x)Q^{-1}$. So $\Phi(x)\boldsymbol{c}$ solves $(*)$ with $\boldsymbol{b}(x) = 0$.
Define *Green's function*

$$G(x,y) := \begin{cases} \Phi(x)B_a\Phi(a)\Phi^{-1}(y) & y \le x, \\ -\Phi(x)B_b\Phi(b)\Phi^{-1}(y) & y > x. \end{cases}$$

Then

$$\underline{\boldsymbol{y}(x)} = \underline{\Phi(x)\boldsymbol{c}} + \int_a^b \underline{G(x,y)\boldsymbol{b}(y)\mathrm{d}y}.$$

$$y_i := \sum_{j=1}^{n} G_{ij} b_j$$

Can verify that this solves $(*)$ by plug'n'chug.

ODE Systems: Conditioning

For perturbed problem with $\boldsymbol{b}(x) + \Delta\boldsymbol{b}(x)$ and $\boldsymbol{c} + \Delta\boldsymbol{c}$:

$$\|\Delta\boldsymbol{y}\|_\infty \le \max\left(\|\Phi\|_\infty, \|G\|_\infty\right)\left(\|\Delta\boldsymbol{c}\|_1 + \int \|\Delta\boldsymbol{b}(y)\|_1 \,\mathrm{d}y\right).$$

- Conditioning bound implies uniqueness.
- Also get continuous dependence on data.

Shooting Method

Idea: Want to make use of the fact that we can already solve IVPs.

Problem: Don't know *all* left BCs.

$y'' = f(y, y')$

$\Rightarrow y(a) =$

$\cancel{y(b) =}$

$\Rightarrow y'(a) = s$

$s \mapsto y_s(b)$

Demo: Shooting method [cleared]

What about systems?

Cannons aim in more than 1D.

What are some downsides of this method?

- Can fail
- Can be unstable even if the ODE is stable

What's an alternative approach?

Write a big linear(-ized) system

Finite Difference Method

Idea: Replace u' and u'' with finite differences.

For example: second-order centered

$$\begin{aligned} u'(x) &= \frac{u(x+h)-u(x-h)}{2h} + O(h^2) \\ u''(x) &= \frac{u(x+h)-2u(x)+u(x-h)}{h^2} + O(h^2) \end{aligned}$$

Demo: Finite differences [cleared]

What happens for a nonlinear ODE?

Linearize, Newton

Demo: Sparse matrices [cleared]

Collocation Method

$$(*)\begin{cases} y'(x) = f(y(x)), \\ g(y(a), y(b)) = 0. \end{cases}$$

1. Pick a basis (for example: Chebyshev polynomials)

$$\hat{y}(x) = \sum_{i=1}^{n} \alpha_i T_i(x)$$

DoF = n

Want $\hat{y}$ to be close to solution y. So: plug into $(*)$.

Problem: $\hat{y}$ won't satisfy the ODE at all points at least. We do not have enough unknowns for that.

2. Idea: Pick n points where we would like $(*)$ to be satisfied.
$\rightarrow$ Get a big (non-)linear system
3. Solve that (LU/Newton)$\rightarrow$ done.

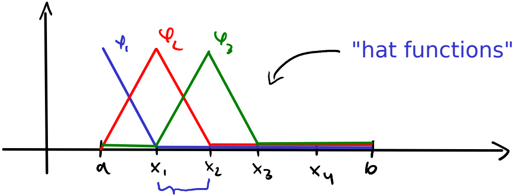
Galerkin/Finite Element Method

$u''(x) - f(x) := R(x)$

$$u''(x) = f(x), \qquad u(a) = u(b) = 0.$$

Problem with collocation: Big dense matrix.

Idea: Use piecewise basis. Maybe it'll be sparse.



"hat functions"

one "finite element"

What's the problem with that?

u' does not exist

u'' 〃 〃 〃

Weak solutions/Weighted Residual Method

Idea: Enforce a 'weaker' version of the ODE.

"moments"

$$\int_a^b u''(x)\psi(x)\,dx = -\int_a^b u'(x)\psi'(x)\,dx + \left[u'(x)\psi(x)\right]_a^b$$

Solve: $\int_a^b u''(x)\psi(x)\,dx = \int_a^b f(x)\psi(x)\,dx$

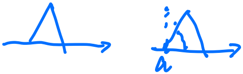
$$\int_a^b [u''(x) - f(x)]\cdot\underline{\psi(x)}\,dx = 0$$

$$\sum r(x_i)\underline{\psi(x_i)w_i} = 0$$

"weighted residual methods"

collocation: $\varphi_i(x) = \delta(x - x_j)$

Galerkin: Choices in Weak Solutions



Make some choices:

- Solve for $u \in \operatorname{span}\{\text{hat functions } \varphi_i\}$
- Choose $\psi \in W = \operatorname{span}\{\text{hat functions } \varphi_i\}$ with $\psi(a) = \psi(b) = 0$.
  $\to$ Kills boundary term $[u'(x)\psi(x)]_a^b$.

These choices are called the Galerkin method. Also works with other bases.

Discrete Galerkin

Assemble a matrix for the Galerkin method.

$$-\int_a^b \underline{\underline{u'(x)}}\,\psi'(x)\,dx = \int_a^b f(x)\,\psi(x)\,dx\ , \quad \psi(x)\in\{\varphi_j(x)\}$$

$$-\int_a^b \sum_{j=1}^{n} \alpha_j \varphi_j'(x)\,\psi'(x)\,dx) = \quad " \quad , \quad \text{let } \psi(x)=\varphi_i(x)$$

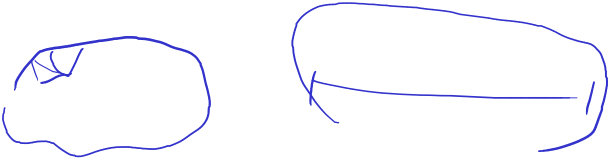
$$-\sum_{j=1}^{n}\Big[\underbrace{\int_a^b \varphi_j'(x)\,\varphi_i'(x)\,dx}_{S_{ij}}\Big]\underbrace{\alpha_j}_{\vec{\alpha}} = \underbrace{\int_a^b f(x)\,\varphi_i(x)\,dx}_{r_i}$$

$$S\vec{\alpha} = \vec{r}$$

$\uparrow$
sparse.

Outline

Advertisement



Remark: Both PDEs and Large Scale Linear Algebra are big topics. Will only scratch the surface here. Want to know more?

- CS555 → Numerical Methods for PDEs ← spring
- CS556 → Iterative and Multigrid Methods ← Fall
- CS554 → Parallel Numerical Algorithms ← fall

We would love to see you there! :)

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Solving Sparse Linear Systems

Solving $A\boldsymbol{x} = \boldsymbol{b}$ has been our bread and butter.

Typical approach: Use factorization (like LU or Cholesky)
Why is this problematic?

Idea: Don't factorize, iterate.
Demo: Sparse Matrix Factorizations and "Fill-In" [cleared]

'Stationary' Iterative Methods

Idea: Invert only part of the matrix in each iteration. Split

$$A = M - N,$$

$\boxed{A}\, x = b$

where M is the part that we are actually inverting. Convergence?

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ M\mathbf{x} &= N\mathbf{x} + \mathbf{b} \\ M\mathbf{x}_{k+1} &= N\mathbf{x}_k + \mathbf{b} \\ \mathbf{x}_{k+1} &= M^{-1}(N\mathbf{x}_k + \mathbf{b}) \end{aligned}$$

- These methods are called *stationary* because they do the same thing in every iteration.
- They carry out fixed point iteration.
  $\rightarrow$ Converge if contractive, i.e. $\rho(M^{-1}N) < 1$.
- Choose M so that it's easy to invert.

Choices in Stationary Iterative Methods

What could we choose for M (so that it's easy to invert)?

Name	M	N
Jacobi	D	$-(L+U)$
Gauss-Seidel	$D+L$	$-U$
SOR	$\frac{1}{\omega}D+L$	$\left(\frac{1}{\omega}-1\right)D-U$

where L is the below-diagonal part of A, and U the above-diagonal.

Demo: Stationary Methods [cleared]