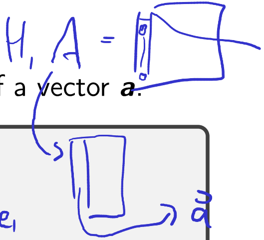
- Examlet 2 : Wed – Fri
  - ↪ material through 9/22 includes QR, LSQ, Gram-Schmidt
- HW 5
  - ↪ $\|Ax - b\| + \|Mx\|$ regularization

$$Ax \cong b \rightsquigarrow A = U\Sigma V^T$$

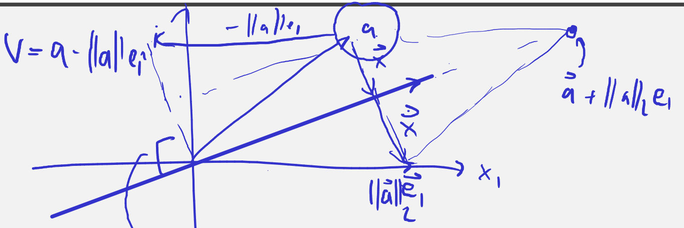
$$\|Ax - b\| = \|U\Sigma V^T x - b\|$$

  - ↪ RQ factorization
    - ↪ GQR
    - ↪ CLSQ

Householder Transformations



Find an *orthogonal* matrix Q to zero out the lower part of a vector $\boldsymbol{a}$.



$$(\vec{a} - \|\vec{a}\| e_1) \cdot (\vec{a} + \|\vec{a}\| e_1) = \|a\|^2 - \|a\|^2 = 0$$

$$\underbrace{a - 2\vec{v}\frac{(\vec{v}^T)a}{\|v\|_2^2}}_{x} = \underbrace{\left(I - 2\frac{vv^T}{v^Tv}\right)}_{\text{Householder reflector.}}\vec{a}$$

Householder Reflectors: Properties

Seen from picture (and easy to see with algebra):

$$H = I - 2\frac{vv^T}{v^Tv}$$

$$H\boldsymbol{a} = \pm \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1.$$

Remarks:

- Q: What if we want to zero out only the $i+1$th through nth entry?
  A: Use $\boldsymbol{e}_i$ above.
- A product $H_n \cdots H_1 A = R$ of Householders makes it easy (and quite efficient!) to build a QR factorization.
- It turns out $\boldsymbol{v}' = \boldsymbol{a} + \|\boldsymbol{a}\|_2 \, \boldsymbol{e}_1$ works out, too–just pick whichever one causes less cancellation.

  $v = a - \|a\|_2 e_1$

  choose so a_1, $\pm \|a\|_2 e_1$ have same sign.
- H is symmetric
- H is orthogonal

Demo: 3x3 Householder demo [cleared]

$$H\vec{a} = \left(I - 2\frac{vv^T}{v^Tv}\right)a = a - 2\frac{v(v^T a)}{v^Tv}$$

$O(n)$ per colum

$\rightarrow HA \rightarrow O(n^2)$

Givens Rotations

If reflections work, can we make rotations work, too?

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \|a\| \\ 0 \end{bmatrix}$$

$$c = a_1 / \|a\| \qquad s = a_2 / \|a\|$$

Demo: 3x3 Givens demo [cleared]

For QR

$G_3 G_2 G_1$

Rank-Deficient Matrices and QR

approximately

What happens with QR for rank-deficient matrices?

$$AP = QR$$

$$AP = Q \begin{pmatrix} \text{big} & & \\ & \ddots & \text{smaller} \\ & & \text{smallest} \end{pmatrix}$$

"pivoted" QR, rank-revealing QR

Rank-Deficient Matrices and Least-Squares

What happens with Least Squares for rank-deficient matrices?

↪ or short/fat

$$A\boldsymbol{x} \cong \boldsymbol{b}$$

- QR finds a solution with minimal $\|Ax-b\|_2$.
- But: not unique, $x+n$ with $n \in N(A)$ is just as good.
- Wish: additional cond: <u>also</u> minimize $\|x\|_2$

↪ use the SVD.

SVD: What's this thing good for? (I)

$A = U \Sigma V^T$

$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix}$

$\|A\|_2 = \sigma_1$

$\operatorname{cond}(A) = \sigma_1 / \sigma_n$

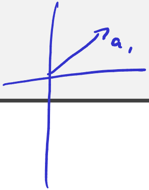
$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

$N(A) = \operatorname{span}(v_{k+1}, \dots, v_n)$

$\operatorname{rank}(A) = \#\{\sigma_i \neq 0\}$ → not sensible numerically

$A = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$

num-rank$(A, \varepsilon) = \#\{\sigma_i > \varepsilon\}$



consequence

definition

SVD: What's this thing good for? (II)

- *Low-rank Approximation*

Theorem (Eckart-Young-Mirsky)

If $k < r = \operatorname{rank}(A)$ *and*

$$A = U \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} V^T$$

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T, \quad \textit{then}$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k}^{n} \sigma_j^2}.$$

Demo: Image compression [cleared]

SVD: What's this thing good for? (III)

- The minimum norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$: