- Site looks broken? Shift + click "reload".
- Ex cancel 2 grades out

Review:

- Eigenvalue problems $AX = X\Lambda$ ($\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$)
- Sensitivity bound: $Ax = \lambda x$ $\quad (A+E)x = \mu x$

$$\max_{\mu} |\mu - \text{"closest } \lambda_k\text{"}| \le \mathrm{cond}(X) \cdot \|E\|$$

A symm. $\Rightarrow$ X orthogonal $\Rightarrow$ $\mathrm{cond}_2(X) = 1$

- Operations
  - $A^k \to \lambda^k$, eigvec same $\quad A^{-1} \to \lambda^{-1}$, eigvec same $\quad A - \sigma I \to \lambda - \sigma$, eigvec same

Power Iteration

Demo: Motivating Power Iteration [cleared]

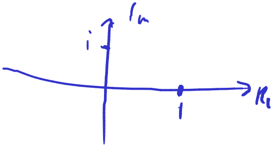
Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Further assume $\|\boldsymbol{x}_i\| = 1$.

Use random vector: $\vec{y}_0 = \alpha \vec{x}_1 + \beta \vec{x}_2$

$$\vec{y}_{1000} = A^{1000} \vec{y}_0 = \alpha \lambda_1^{1000} \vec{x}_1 + \beta \lambda_2^{1000} \vec{x}_2$$

$$\frac{\vec{y}_{1000}}{\lambda_1^{1000}} = \frac{A^{1000} \vec{y}_0}{\lambda_1^{1000}} = \alpha \underbrace{\cancel{\frac{\lambda_1^{1000}}{\lambda_1^{1000}}}}_{1} \vec{x}_1 + \beta \underbrace{\frac{\lambda_2^{1000}}{\lambda_1^{1000}}}_{\underbrace{\left(\frac{\lambda_2}{\lambda_1}\right)^{1000}}_{|\cdot|<1}} \vec{x}_2$$

Power Iteration: Issues?



What could go wrong with Power Iteration?

- $|\lambda_2| = |\lambda_1|$? (includes multiplicities)
- Overflow $\to$ "normalized power iteration"
- $|\lambda_2| \approx |\lambda_1| \Rightarrow$ conv. factor $\left|\frac{\lambda_2}{\lambda_1}\right| \approx 1$

  $\Rightarrow$ need lots of iterations

- only get the first one
- what if $\alpha = 0$?
  - $\to$ actual problem in exact arith.
  - $\to$ maybe OK w/ rounding?
- λ complex?

What about Eigenvalues?

$$\frac{(Ax)_1}{x_1} = \lambda$$

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

$$\frac{x^T A \vec{x}}{x^T \vec{x}}$$ "Rayleigh quotient"

$A\vec{x} = \lambda\vec{x} \Rightarrow$ $\hookrightarrow = \lambda$

Convergence of Power Iteration

What can you say about the convergence of the power method?

Say $\boldsymbol{v}_1^{(k)}$ is the kth estimate of the eigenvector $\boldsymbol{x}_1$, and

$$e_k = \left\| \boldsymbol{x}_1 - \boldsymbol{v}_1^{(k)} \right\|.$$

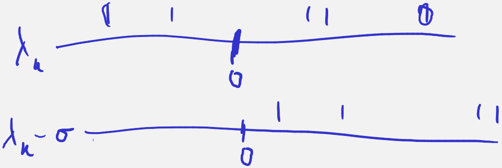
$$v_1^{(k+1)} = A\, v_1^{(k)}$$

$$e_{k+1} \approx \left|\frac{\lambda_2}{\lambda_1}\right| e_k$$

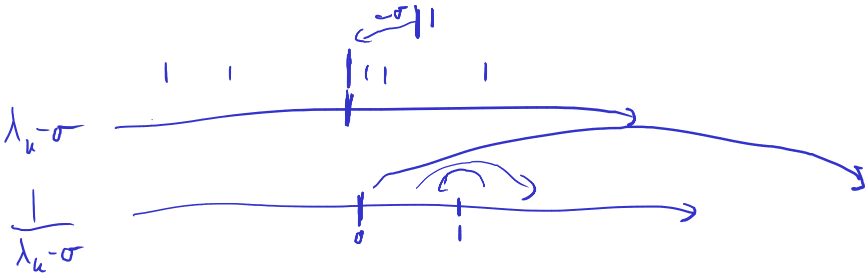
linearly convergent

(quadratically conv

$e_{k+1} \approx c \cdot e_k^2$)



$$\left|\frac{\lambda_2 - \sigma}{\lambda_1 - \sigma}\right|$$



Shift-invert

Inverse Iteration

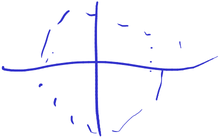
shifted: $\left|\frac{\lambda_2' - \sigma}{\lambda_1' - \sigma}\right|$

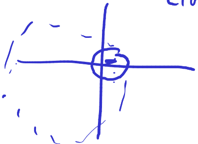
invert: $\left|\frac{\lambda_1'' -}{\lambda_2'' - \sigma}\right|$

Describe *inverse iteration*.

$$\vec{x}_{k+1} = (A - \sigma I)^{-1} x_k$$

$$e_{k+1} \approx \left|\frac{\lambda_{\text{closest to }\sigma} - \sigma}{\lambda_{\text{second-to-closest-to-}\sigma} - \sigma}\right| e_k$$





complex shifts: also OK.

Rayleigh Quotient Iteration

Inverse it: $x_{k+1} = (A - \sigma I)^{-1} x_k$

n^3 setup

n^2 per it. solve $Ax = b$

$\Leftrightarrow x = A^{-1} b$

Describe *Rayleigh Quotient Iteration*.

0 setup

n^3 per it.

$$\sigma_k = \frac{\vec{x}_k^T A \vec{x}_k}{\vec{x}_k^T \vec{x}_k}$$

$$\vec{x}_{k+1} = (A - \sigma_k I)^{-1} \vec{x}_k$$

Demo: Power Iteration and its Variants [cleared]

In-Class Activity: Eigenvalues

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