- Examlet 3
- HW 8

Solving Nonlinear Equations

What is the goal here?

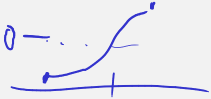
$$f(\vec{x}) = \vec{0}$$

↳ 1D case: $f(x) = 0$

Showing Existence

How can we show existence of a root?

- Intermediate value theorem: continuous,



- Inverse function theorem: J_f invertible at some point $x \in \mathbb{R}^n$ $\Rightarrow$ there exists $\varepsilon > 0$ so that f is invertible on $B(x, \varepsilon)$

Want: $f^{-1}(\vec{0}) \leftarrow$ goal

- Contraction mapping theorem

$g(x) = f(x) + x$

A function $g: \mathbb{R}^n \to \mathbb{R}^n$ is called contractive if there exists $0 < \gamma < 1$ so that $\|g(x) - g(y)\| \le \gamma \|x - y\|$.

On a closed set $S \subseteq \mathbb{R}^n$ with $g(S) \subseteq S$ there exists a unique fixed point x^*: $g(x^*) = x^*$.

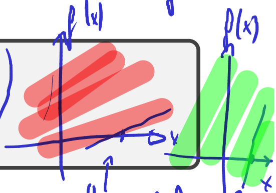
Sensitivity and Multiplicity

abs cond of evaluating f^{-1}

f'

What is the sensitivity/conditioning of root finding?

$$\underset{\text{abs}}{\text{cond}}(\text{root finding}) = \underset{\text{abs}}{\text{cond}}\left(f^{-1}(\vec{0})\right)$$



well-cond f
poorly cond f^{-1}

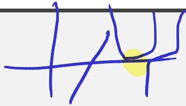
poorly f
well-cond f^{-1}

What are multiple roots?

Example: $f(x) = (x - x_*)^2 \cdot \tilde{f}(x)$

Def: $f(x^*) = 0,\ f'(x^*) = 0,\ \ldots$

multiplicity 2



How do multiple roots interact with conditioning?

inverse is steep, therefore conditioning is poor.

Rates of Convergence

What is *linear convergence*? *quadratic convergence*?

$$\vec{e}_k = \hat{\vec{u}}_k - \vec{u}$$

$\hat{\vec{u}}_k$: guess at k'th iteration

$\vec{u}$: true answer

$\vec{e}_k$: error

An iterative method converges with rate r iff

$$\lim_{k \to \infty} \frac{\|\vec{e}_{k+1}\|}{\|\vec{e}_k\|^r} = C \quad \begin{matrix} > 0 \\ < \infty \end{matrix}$$

$r=1$: linear conv

$r=2$: quadratic

$r>1$: superlinear

Drop lim? Cartoon def: $\|e_{k+1}\| \le C \cdot \|e_k\|^r$

$$\|\vec{e}_{k+1}\| \le C \cdot \|\vec{e}_k\|$$

$$\|e_{k+1}\| \le C \cdot \|e_k\|^2$$

About Convergence Rates

Demo: Rates of Convergence [cleared]

Characterize linear, quadratic convergence in terms of the 'number of accurate digits'.

Linear: gains fixed number of digits per it.

Quadratic: doubles number of digits.

Stopping Criteria

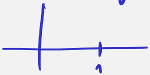
Comment on the 'foolproof-ness' of these stopping criteria:

1. $|f(x)| < \varepsilon$ ('residual is small')
2. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < \varepsilon$
3. $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| \,/\, \|\boldsymbol{x}_k\| < \varepsilon$

1

long ways from the root. $\Rightarrow$ fail

2. Method gets stuck $\Rightarrow$ fail



: rel accuracy of roots with diff. magnitude not the same:

3. Goes bad if $\|\vec{x}_k\|$ is small

Bisection Method

Demo: Bisection Method [cleared]

What's the rate of convergence? What's the constant?

linear

Fixed Point Iteration

$$\begin{aligned} x_0 &= \langle \text{starting guess} \rangle \\ x_{k+1} &= g(x_k) \end{aligned}$$

Demo: Fixed point iteration [cleared]

When does fixed point iteration converge? Assume g is smooth.

Let x^* be the fixed pt. with $g(x^*) = x^*$

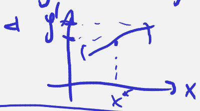
If $|g'(x^*)| < 1$, then there exists a neighborhood where we have convergence.

$$e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$$

Fixed Point Iteration: Convergence cont'd.

Error in FPI: $e_{k+1} = x_{k+1} - x^* = g(x_k) - g(x^*)$

$$e_{k+1} = g(x_k) - g(x^*) = g'(\theta_k)(x_k - x^*) = \underbrace{g'(\theta_k)}\, e_k$$

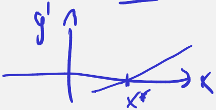


$\theta_k \in [x_k, x^*]$

if close enough to x^* that $|g'(\theta_k)| < 1$ $\Rightarrow$ conv.

What if $g'(x^*) = 0$?

Does <u>not</u> imply that $g'(\theta_k) = 0$.



$|g'(x^* + h)| \approx c|h|$

$|h| \approx |e_k|$

$\Rightarrow |e_{k+1}| \approx C \cdot h \cdot |e_k| \approx C \cdot |e_k|^2$

Newton's Method

$f(x)=0$?!

Derive Newton's method.

$$f(x_k+h) \approx f(x_k) + f'(x_k)\,h + \cancel{O(h^2)}$$

Solve $f(x_k)+f'(x_k)\,h = 0$ for h.

Use $x_{k+1} = x_k + h$.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = g(x_k)$$

Demo: Newton's method [cleared]

Convergence and Properties of Newton

What's the rate of convergence of Newton's method?

Drawbacks of Newton?

Demo: Convergence of Newton's Method [cleared]