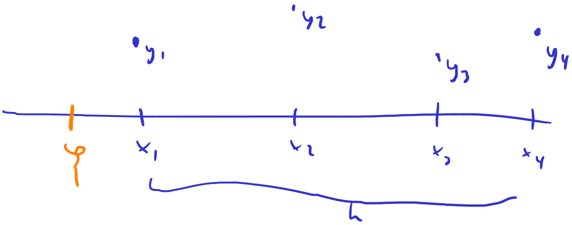
- EC HW 14 out later today



$$f'(\xi) \approx \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4 + O(h^?)$$

$$\alpha_i = \Psi_i(x_1, \ldots, x_4)$$

Differentiation Matrices

How can numerical differentiation be cast as a matrix-vector operation?

$$p_{n-1}(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x)$$

$$V\vec{\alpha} = \vec{y} \iff \vec{\alpha} = V^{-1}\underbrace{\vec{y}}_{f(\vec{x})}$$

$$p'_{n-1}(x) = \sum_{i=1}^{n} \alpha_i \varphi'_i(x)$$

$$V' = \begin{pmatrix} \varphi'_1(x_1) & \varphi'_2(x_1) & \cdots \\ \varphi'_1(x_2) & & \\ & & \end{pmatrix}$$

$$f'(\vec{x}) \approx \overbrace{V'\,V^{-1}}^{D} f(\vec{x})$$

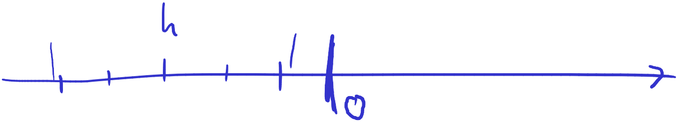
Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Build D)

$$\begin{pmatrix} f'(x_1) \\ \vdots \\ f'(x_n) \end{pmatrix} = \underbrace{\boxed{\quad D \quad}}_{V'V^{-1}} \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$$f'(\xi) \approx \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 + \alpha_4 y_4 + O(h^?)$$

$\Rightarrow$ Each row of D contains a finite difference rule

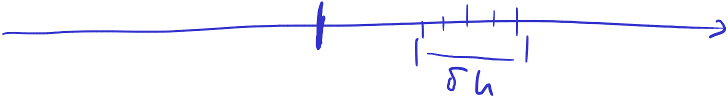
$$f'(x_1) \approx d_{11} f(x_1) + \cdots + d_{1n} f(x_n)$$



Shift:

→ does not change D, does not chg FD rule

Scale:



$$f'(0) \approx -\frac{1}{h} f(0) + \frac{1}{h} f(h)$$

$$= \frac{f(h) - f(0)}{h}$$

→ scaling nodes by δ: become $\frac{\vec{\alpha}}{\delta}$

Properties of Differentiation Matrices

How do I find second derivatives?

Does D have a nullspace?

Numerical Differentiation: Shift and Scale

Does D change if we shift the nodes $(x_i)_{i=1}^n \to (x_i + c)_{i=1}^n$?

Does D change if we scale the nodes $(x_i)_{i=1}^n \to (\alpha x_i)_{i=1}^n$?

Finite Difference Formulas from Diff. Matrices

How do the rows of a differentiation matrix relate to FD formulas?

Assume a large equispaced grid and 3 nodes w/same spacing. How to use?

Finite Differences: via Taylor

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} + O(h^1)$$

$$f(x+h) = f(x) + f'(x)\,h + f''(x)\frac{h^2}{2} + \dots$$

$$\frac{\cancel{f(x)} + f'(x)\,h + f''(x)\frac{h^2}{2} + \dots - \cancel{f(x)}}{h} = f'(x) + O(h)$$

More Finite Difference Rules

Similarly:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

(Centered differences)

Can also take higher order derivatives: → using deriv. matrices! D^2

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Can find these by trying to match Taylor terms.
Alternative: Use linear algebra with interpolate-then-differentiate to find FD formulas.

Demo: Finite Differences vs Noise [cleared]

Demo: Floating point vs Finite Differences [cleared]

Outline

What can we solve already?

- Linear Systems: yes
- Nonlinear systems: yes
- Systems with derivatives: no

Some Applications

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : [0, \overset{t}{\infty}) \to \mathbb{R}^2$

$t=0$ $\qquad$ $t=1$

IVPs	BVPs
▶ Population dynamics $y_1' = \underline{y_1}(\underline{\alpha_1} - \beta_1 \underline{y_2})$ (prey) $y_2' = y_2(-\underline{\alpha_2} + \underline{\beta_2 y_1})$ (predator) ▶ chemical reactions ▶ equations of motion	▶ bridge load ▶ pollutant concentration (steady state) ▶ temperature (steady state) ▶ waves (time-harmonic)

Demo: Predator-Prey System [cleared]

Initial Value Problems: Problem Statement

Want: Function $\boldsymbol{y} : [0, T] \to \mathbb{R}^n$ so that

- $\boldsymbol{y}^{(k)}(t) = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k-1)})$ (*explicit*), or
- $\boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k)}) = 0$ (*implicit*)

are called explicit/implicit *kth-order ordinary differential equations* (*ODEs*). Give a simple example.

$y' = \alpha y \qquad y(t) = c \cdot e^{\alpha t}$

Not uniquely solvable on its own. What else is needed?

$y'' = f(y, y')$ → needs y and y'

need $\left.\begin{array}{l} y(0) = \ldots \\ \vdots \\ y^{(k-1)}(0) = \ldots \end{array}\right\}$ k initial conditions

Reducing ODEs to First-Order Form

$\vec{w}' = f(\vec{w})$

A kth order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y)$$

$$\boxed{\begin{aligned} w_1 &= y \\ w_2 &= y' \end{aligned}}$$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}' = \begin{bmatrix} w_2 \\ f(w_1) \end{bmatrix}$$

→ For numerics: need only worry about (systems of) first-order ODEs

Properties of ODEs

$$\vec{y}' = \vec{f}(t, \vec{y})$$

What is a linear ODE?

$$\vec{f}(t, \vec{y}) = A(t)\,\vec{y} + \vec{b}(t)$$

What is a linear and homogeneous ODE?

$$\vec{f}(t, \vec{y}) = A(t)\,\vec{y}$$

What is a constant-coefficient ODE?

$$f(t, \vec{y}) = A\vec{y} + \vec{b}$$

Existence and Uniqueness

Consider the perturbed problem

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases} \qquad \begin{cases} \widehat{\boldsymbol{y}}'(t) = \boldsymbol{f}(\widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}}(t_0) = \widehat{\boldsymbol{y}}_0 \end{cases}$$

$y' = \alpha y \to e^{\alpha y}$

$f(y) = \alpha y$

$f'(y) = \alpha$

Then if $\boldsymbol{f}$ is *Lipschitz continuous* (has 'bounded slope'), i.e.

$$\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\widehat{\boldsymbol{y}})\| \le L \|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|$$

(where L is called the *Lipschitz constant*), then...

- there exists a solution y in a neighborhood of t_0
- $\|\vec{y}(t) - \vec{\hat{y}}(t)\| \le e^{L(t-t_0)} \|\vec{y}_0 - \vec{\tilde{y}}_0\|$

Picard-Lindelöf theorem

What does this mean for uniqueness?

Implicitly covers uniqueness as well

$\vec{y}_0 = \tilde{y}_0 \Rightarrow \vec{y}(t) = \tilde{y}(t)$

Conditioning

Unfortunate terminology accident: "Stability" in ODE-speak

To adapt to conventional terminology, we will use 'Stability' for

- the conditioning of the IVP, *and*
- the stability of the methods we cook up.

Some terminology:

An ~~ODE is~~ IVP stable if and only if...

The solution is continuously dependent on the IC.
For all $\varepsilon > 0$ there exists a $\delta > 0$ so that
$\|\hat{\vec{y}}_0 - \vec{y}_0\| < \delta \Rightarrow \|\hat{\vec{y}}(t) - \vec{y}(t)\| < \varepsilon$ for all $t \geq t_0$.

An ~~ODE~~ IVP is asymptotically stable if and only if

$\|\hat{\vec{y}}(t) - \vec{y}(t)\| \to 0 \quad t \to \infty$

Example I: Scalar, Constant-Coefficient

$$\begin{cases} y'(t) = \lambda y \\ y(0) = y_0 \end{cases} \quad \text{where} \quad \lambda = a + ib$$

Solution?

magnitude.

$$y(t) = y_0 e^{\lambda t} = y_0 e^{at} e^{ibt}$$

$|\cdot| \le 1$ "oscillation"

When is this stable?

$\operatorname{Re} \lambda > 0$

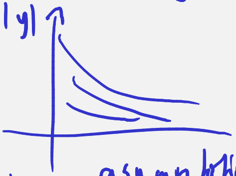
unstable

$\operatorname{Re} \lambda = 0$

stable

not asymp. stable

$\operatorname{Re} \lambda < 0$



asymptotically stable

Example II: Constant-Coefficient System

hom. const. coeff.

$A = VDV^{-1}$

$$\begin{cases} \boldsymbol{y}'(t) = A\boldsymbol{y}(t) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

Assume $V^{-1}AV = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ diagonal. Find a solution.

$\vec{w} = V^{-1}\vec{y} \quad \rightsquigarrow \quad V\vec{w} = \vec{y}$

$w' = V^{-1}y' = V^{-1}Ay = \cancel{V^{-1}V} D V^{-1} y$

$= D w$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}' = \begin{matrix} d_{11} w_1 \\ \vdots \\ d_{nn} w_n \end{matrix}$

When is this stable?

$\operatorname{Re}(\lambda_i) < 0 \to$ asympt. stable

$> 0 \to$ unstable.

Euler's Method

Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

diff eq

- Discrete times: $t_1, t_2, \ldots$, with $t_{i+1} = t_i + h$
- Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

int eq.

$$\vec{y}(t) = \vec{y}_0 + \int_{t_0}^{t} f(y(\tau))\, d\tau$$

Picard's int. eq.

use quadrature.

Forward Euler: $y(1) = y_0 + f(y_0) \cdot \Delta t$

"left rectangle rule": $\int_a^b f(x)\,dx \approx f(a) \cdot (b-a)$

Euler's method: Forward and Backward

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(\tau))\mathrm{d}\tau,$$

Use 'left rectangle rule' on integral:

$$\vec{y}_{k+1} = \underbrace{\vec{y}_k + h f(y_k)}_{\text{explicit} \to \text{can just evaluate } y_{k+1}}$$

Use 'right rectangle rule' on integral:

$$\underbrace{\underline{\vec{y}_{k+1}} = \vec{y}_k + h f(\underline{y_{k+1}})}_{\text{implicit} \to \text{solve for } y_{k+1}}$$

Demo: Forward Euler stability [cleared]