- HW 14 (ec, due tomorrow)
- 4CH 2 (+5% if in by tomorrow, otherwise during finals)
- Final (Dec 8—12)
- ICES (please fill them out!)

Euler's Method

Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

- Discrete times: $t_1, t_2, \ldots$, with $t_{i+1} = t_i + h$
- Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau))\, d\tau$$

Euler's method: Forward and Backward

$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(\tau))\mathrm{d}\tau,$$

Use 'left rectangle rule' on integral:

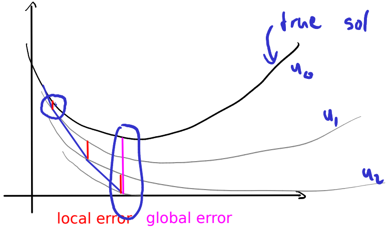
$\vec{y}_{k+1} = \vec{y}_k + h f(\vec{y}_k)$ ✓ Fw Euler

Use 'right rectangle rule' on integral:

solve for y_{k+1} $\vec{y}_{k+1} = \vec{y}_k + h f(\vec{y}_{k+1})$ Bw Euler

Demo: Forward Euler stability [cleared]

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$. Define the local error at step k as...

$$\ell_k = y_k - u_{k-1}(t_k)$$

Define the global error at step k as...

$$g_k = y(t_k) - y_k$$

About Local and Global Error

Is global error $= \sum$ local errors?

No, just like compound interest

A time integrator is said to be *accurate of order p* if...

$\ell_n = O(h^{p+1})$

ODE IVP Solvers: Order of Accuracy

A time integrator is said to be *accurate of order p* if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect–why?

$$\underbrace{\frac{1}{h}}_{\text{\# steps to get to time } O(1)} \cdot O(h^{p+1}) = O(h^p)$$

for sum of local errors

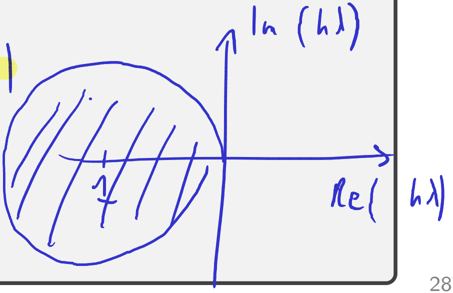
Stability of a Method

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$
\begin{aligned}
y_{k+1} &= y_k + h\lambda y_k \\
&= y_k (1 + h\lambda) \\
&= (1 + h\lambda)^{k+1} y_0
\end{aligned}
$$

stable $\iff$ $|1 + h\lambda| \leq 1$

stability region for Fw Euler



Stability: Systems

$A = V^{-1} D V$

$w = V \vec{y}$

What about stability for systems, i.e. $D = V A V^{-1}$

$y_{k+1} = y_k + A y_k$ $\quad \mathbf{y}'(t) = A\mathbf{y}(t)$?

$w_i' = \lambda_i w_i$

- Diagonalize the system $\quad \vec{w}' = D \vec{w}$

$w_{k+1} = V y_{k+1} = V(y_k + A y_k)$

$= V y_k + V A y_k$

$= w_k + V V^{-1} D V y_k$

$w' = V y'$

$= V A y$

$= V V^{-1} D V y$

$= D w$

Stability: Nonlinear ODEs

$\hookrightarrow$ only eigenvalues matter.

What about stability for nonlinear systems, i.e.

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t))?$$

$$e(t) = y(t) - \hat{y}(t)$$

$$e'(t) = f(y(t)) - f(\hat{y}(t)) \simeq J_F(y)\, e(t)$$

$+$ HOT

$$\vec{e}\,' = J_F \vec{e}$$

vs. stab. region

$\rightarrow$ look at eigenvalues of Jacobian of f

Stability for Backward Euler

Find out when backward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_{k+1} = y_k + h\lambda y_{k+1}$$

$$y_{k+1}(1 - h\lambda) = y_k$$

$$y_{k+1} = \frac{1}{1-h\lambda} y_k = \left(\frac{1}{1-h\lambda}\right)^{k+1} y_0$$

$$\text{stable} \iff \left|\frac{1}{1-h\lambda}\right| \le 1$$

Demo: Backward Euler stability [cleared]

Stiff ODEs: Demo

Demo: Stiffness [cleared]

$\mathrm{Re}(\lambda) \geq 0$

$\rightarrow$ ODE unstable

$\mathrm{Re}(\lambda) < 0$ (ODE asymp stable)

$\Rightarrow$ pick any time step, stable anyway

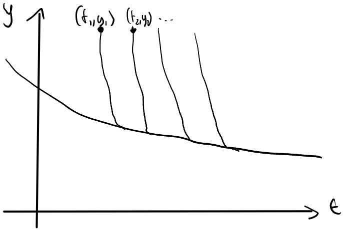
If a method is stable for all h when $\mathrm{Re}(\lambda) \leq 0$:
"A-stable"

$$y' = -100y + 100t + 101$$

$$y_{k+1} = y_k + h\left(-100y_{k+1} + 100t + 101\right)$$

$$(1 + 100h)\, y_{k+1} = y_k + h\left(100t + 101\right)$$

'Stiff' ODEs



- Stiff problems have *multiple time scales*.
  (In the example above: Fast decay, slow evolution.)
- In the case of a stable ODE system

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)),$$

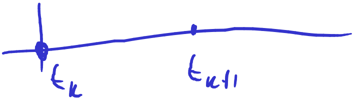
stiffness can arise if $J_{\boldsymbol{f}}$ has eigenvalues of very different magnitude.

Stiffness: Observations

Why not just ‘small’ or ‘large’ magnitude?

What is the problem with applying explicit methods to stiff problems?

Predictor-Corrector Methods



Idea: Obtain intermediate result, improve it (with same or different method).

For example:

"predictor" $\tilde{y}_{n+1} = y_n + h f(y_n)$

"corrector": $y_{n+1} = y_n + \frac{h}{2}\left(f(y_n) + f(\tilde{y}_n)\right)$

Heun's method

$\rightarrow$ 2nd order global accuracy

Runge-Kutta/'Single-step'/'Multi-Stage' Methods

Idea: Compute intermediate 'stage values', compute new state from those:

solve
$$\begin{cases} r_1 = f(t_n + c_1 h, y_n + h(a_{11} r_1 + \cdots + a_{1s} r_s)) \\ \vdots \\ r_s = f(t_n + c_s h, y_n + h(a_{s1} r_1 + \cdots + a_{ss} r_s)) \end{cases}$$
$$y_{n+1} = y_n + h(b_1 r_1 + \cdots + b_s r_s)$$

Can summarize in a *Butcher tableau*:

stages $\downarrow$

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \\ c_s & a_{s1} & \cdots & \\ \hline & b_1 & \cdots & b_s \end{array}$$

Runge-Kutta: Properties

When is an RK method explicit?

nonzeros only below the diagonal

When is it implicit?

otherwise

When is it *diagonally implicit*? (And what does that mean?)

nonzeros not above the diagonal

→ can solve one at a time.

Heun and Butcher

Stuff Heun's method into a Butcher tableau:

1. $\tilde{y}_{k+1} = y_k + hf(y_k)$
2. $y_{k+1} = y_k + \frac{h}{2}(f(y_k) + f(\tilde{y}_{k+1}))$.

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

RK4

What is RK4?

$$
\begin{array}{c|cccc}
0 & 0 & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \multicolumn{2}{c}{\underbrace{\frac{1}{3} \quad \frac{1}{3}}_{\frac{4}{6}}} & \frac{1}{6}
\end{array}
$$

Demo: Dissipation in Runge-Kutta Methods [cleared]

Stability Regions

Why does the idea of stability regions still apply to more complex time integrators (e.g. RK?)

Demo: Stability regions [cleared]

More Advanced Methods

Discuss:

- What is a good cost metric for time integrators?
- AB3 vs RK4
- Runge-Kutta-Chebyshev
- LSERK and AB34
- IMEX and multi-rate
- Parallel-in-time ("Parareal")