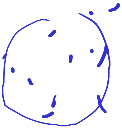
CS 450

- Quiz 2
  - HW1
- Sign into forum at least once to create account
- Categorize forum posts

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 11 & 12 & 13 & 14 \\ & & & \end{pmatrix} 2^{10}$$



$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

(4,)

(4, 1)

(4,)

(4, 4)

rows # cols

1(4,)

(4, 1)

(3, 1)

(4, 1)

Goals:

- numpy bcast
- norms
- "errors"
  - ↳ Fw error / bw error / conditioning
- numbers

Norms: Examples

rel. error (number / scalar) $= \dfrac{|x-\hat{x}|}{|x|}$

rel. error (vector) $= \dfrac{\|x-\hat{x}\|}{\|x\|}$

Examples of norms?

$$\left\|\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

p-norms

∞-norm takes max absolute value

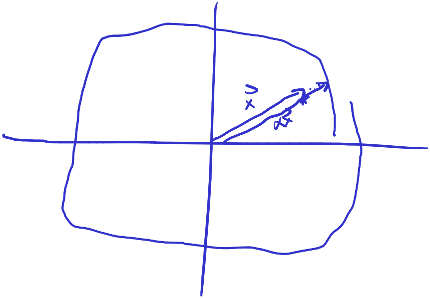
$\left.\begin{array}{l} p \geq 1 \\ p = \infty \end{array}\right\}$ ok

Demo: Vector Norms [cleared]

Unit ball of $\|\cdot\|$:

$$\left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$$

$$\{\vec{x} : \|\vec{x}\| = 1\}$$



$\alpha\vec{x}$

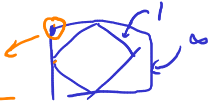
$\|\alpha\vec{x}\| = 1$

$\|\vec{x}\| = \frac{1}{|\alpha|}$

Norms: Which one?

$(-1,1)$

$\left\|\begin{pmatrix}-1\\1\end{pmatrix}\right\|_1 = \sqrt{|-1|+|1|} = 2$



$\|x\|_1 \le 2\, \|x\|_\infty$

2 1

Does the choice of norm really matter much?

In finite-d, all norms are equivalent.

$\|\cdot\|, \|\cdot\|^*$,

$\exists \alpha, \beta \in \mathbb{R}^+, \forall x \in \mathbb{R}^n:$ $\alpha \|x\| \le \|x\|^* \le \beta \|x\|$

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

abs error $= \| \vec{x} - \vec{\tilde{x}} \|$

abs error $\neq$ $\|x\| - \|\tilde{x}\|$

Forward/Backward Error

out   in

Suppose *want* to compute $y = f(x)$, but *approximate* $\hat{y} = \hat{f}(x)$.

What are the forward error and the backward error?

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "\hat{f}"] & f(x) \\
\hat{x} \arrow[r, "f"'] & \hat{f}(x)
\end{tikzcd}$$

fw. error.

Bw. error.

Find the $\hat{x}$ closest to x so that
$f(\hat{x}) = \hat{f}(x)$

$\Delta x = x - \hat{x}$

Forward/Backward Error: Example

$\sqrt{}$

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the forward error?

$$|\Delta y| = |1.4 - 1.4121| = 0.0142\ldots$$

Rel. Fwd. error

$$\frac{|\Delta y|}{|y|} = \frac{0.01\ldots}{1.414\ldots} \approx 0.01$$

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the backward error?

Find $\hat{x}$ s.t. $\sqrt{\hat{x}} = 1.4$ $\hat{x} = 1.96$

Backward error:

$$|\Delta x| = |1.96 - 2| = 0.04$$

Rel. bwd.

$$\frac{|\Delta x|}{|x|} \approx 0.02$$

Forward/Backward Error: Observations

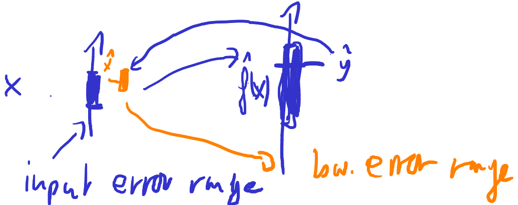
What do you observe about the relative manitude of the relative errors?

Forward/Backward Error: Observations

What do you observe about the relative manitude of the relative errors?

- In this case: Got smaller, i.e. variation *damped out*.
- Typically: Not that lucky: Input error *amplified*.
- If backward error is smaller than the input error: result "as good as possible".

This amplification factor seems worth studying in more detail.



Sensitivity and Conditioning

Consider a more general setting: An input x and its perturbation $\hat{x}$.

$$\frac{|f(x)-f(\hat{x})|}{|f(x)|} \le \kappa_{rel} \frac{|x-\hat{x}|}{|x|}$$

forward/output error   input/bw. error.

If such a factor exists, it is called the (relative) condition number

$$\kappa_{rel} = \max_{(x,\hat{x})\in S} \frac{|f(x)-f(\hat{x})|}{|\,(x)|} \Bigg/ \frac{|x-\hat{x}|}{|x|}$$

$\to$ sup

Absolute Condition Number

Can you also define an *absolute* condition number?