IEI

HW 7

Cold

Goals.

- LSQ: relax "full rank" assumption
- More SVDs! :)
- Eigenvalues

Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

- zeros on diag of R
- $AP = QR = Q$
  big
  smaller
  small
  0?
  0
  rank
- column-pivoted QR ("CPQR")
- rank-revealing QR

Rank-Deficient Matrices and QR

What happens with QR for rank-deficient matrices?

$A = QR$, where R has some ~~small~~ zero diagonal entries, in undetermined order.

Practically, it makes sense to ask for all these 'small' columns to be gathered near the 'right' of R → Column pivoting.

Q: What does the resulting factorization look like?

$$AP = QR$$

$$AP = Q \begin{bmatrix} * & * & * \\ & \text{(small)} & \text{(small)} \\ & & \text{(smaller)} \end{bmatrix}$$

Also used as the basis for *rank-revealing QR*.

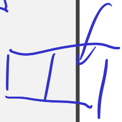
Rank-Deficient Matrices and Least-Squares

What happens with Least Squares for rank-deficient matrices?

$$A\boldsymbol{x} \cong \boldsymbol{b}$$

lin. dep.

TS: 

ST: 

- solution not unique

  [CA flash back: rank-nullity theorem

  $\#\text{cols} = \text{rank} + \dim(\mathcal{N}(A))$

  There exists a nullspace, $\dim(\mathcal{N}(A)) > 0$.

  Suppose $\vec{x}$ minimizes $\|Ax-b\|_2^2$: Let $\vec{n} \in \mathcal{N}(A) \setminus \{\vec{0}\}$.

  What about $\vec{x} + \alpha\vec{n}$?

  $A(\vec{x} + \alpha\vec{n}) = A\vec{x}$ $\Rightarrow$ residual stays the same!

- not unique: ask for extra conditions.

  $\|Ax-b\|_2 \to \min$ + $\|x\|_2 \to \min$

how to solve? $\to$ SVD

SVD: Reduced and Full

economy

For a matrix of shape $m \times n$ with $m > n$, what are the shapes of the factors in the SVD?

TS $\quad m > n$

A: $m \times n$
U: $m \times m$
Σ: $m \times n$
V^T: $n \times n$

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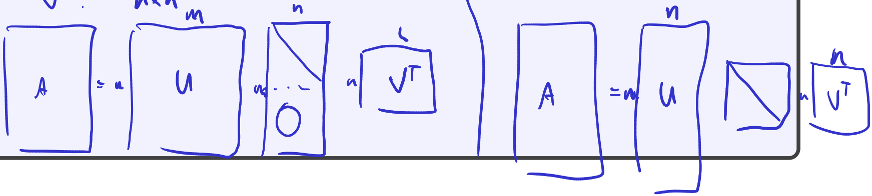


SVD: Reduced and Full

For a matrix of shape $m \times n$ with $m > n$, what are the shapes of the factors in the SVD?

Again, there is the full version of the factorization:

- U: $m \times m$
- Σ: $m \times n$
- V: $n \times n$

and the economical/reduced version:

- U: $m \times n$
- Σ: $n \times n$
- V: $n \times n$

SVD: What's this thing good for? (I)

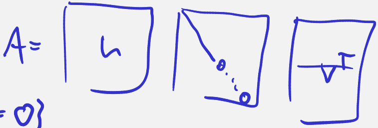
$\|A\|_2 = \sigma_1$

$\kappa_2(A) = \sigma_1 / \sigma_n$

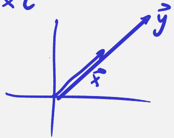
$\text{nullspace}(A) = \text{span}\{v_i : \sigma_i = 0\}$

$\text{rank}(A) = \#\{\sigma_i \neq 0\}$

not computable due to rounding error



for example: 2×2



$\text{numrank}(A, \varepsilon) = \#\{\sigma_i > \varepsilon\}$

CA review:

v_i orthonormal basis

$$v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$\vec{x} = \sum \underbrace{(\vec{v}_i^T \vec{x})}_{\alpha_i} \vec{v}_i$$

$$V^T \vec{x} = \vec{\alpha}$$

$$\underbrace{V^T \vec{x}}_{\vec{\alpha}}$$

SVD: What's this thing good for? (II)

- *Low-rank Approximation*

$$A = \begin{bmatrix} U \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma \end{bmatrix} \begin{bmatrix} V^T \end{bmatrix}$$

Theorem (Eckart-Young-Mirsky)

If $k < r = \operatorname{rank}(A)$ *and*

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T, \quad \textit{then}$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1},$$

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{j=k+1}^{n} \sigma_j^2}.$$

Demo: Image compression [cleared]

$u(x,y) = U_0(x)\, U_1(y)$

Sep. of variables

Use canned basis

↳ sin/cos?

SVD: What's this thing good for? (III)

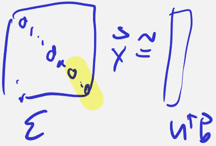
- The minimum norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$:

$$\min \leftarrow \|A\vec{x}-\vec{b}\|_2$$

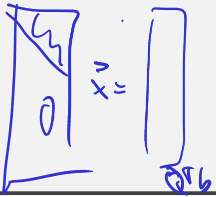
$$= \|U\Sigma V^T\vec{x}-\vec{b}\|_2$$

$$= \|\underbrace{\Sigma V^T\vec{x}}_{\vec{y}:=} - U^T b\|_2$$

$$= \|\Sigma\vec{y} - U^T b\|$$



Recall QR



$y_{k+1} \cdots y_n$ do not matter.

$y_1 \dots y_n$ are uniquely determined. $y_i = \frac{(U^T\vec{b})_i}{\sigma_i}$

<u>Idea:</u> Choose $y_{r+1}, \dots, y_n = 0$

Then $\|\vec{y}\|$ is minimized among all possible choices for $\vec{y}$.

$\vec{y} = V^T \vec{x}$

$V\vec{y} = \vec{x} \Rightarrow \|\vec{x}\|_2 = \|\vec{y}\|_2$

$\Rightarrow \|\vec{x}\|_2$ is also minimized

SVD: Minimum-Norm, Pseudoinverse

$A = U \Sigma V^T$

$A^{-1} = V \Sigma^{-1} U^T$

What is the minimum 2-norm solution to $A\boldsymbol{x} \cong \boldsymbol{b}$ and why?

define pseudo inv. for diag + not full rank

$\Sigma^+ = \mathrm{diag}\begin{pmatrix} 1/\sigma_i & \sigma_i \neq 0 \text{ or some tol} \\ 0 & \text{otherwise} \end{pmatrix}$

def. pseudoinv for not full rank.

$A^+ = V \Sigma^+ U^T$

To solve $A\vec{x} \cong \vec{b}$

$\vec{x} = A^+ \vec{b}$

Generalize the pseudoinverse to the case of a rank-deficient matrix.

in full-rank case: coincides with pseudo inv from normal eqns

Comparing the Methods

Methods to solve least squares with A an $m \times n$ matrix:

Demo: Relative cost of matrix factorizations [cleared]

Comparing the Methods

Methods to solve least squares with A an $m \times n$ matrix:

- Form: $A^T A$: $n^2 m/2$ (symmetric—only need to fill half)
  Solve with $A^T A$: $n^3/6$ (Cholesky)
- Solve with Householder: $mn^2 - n^3/3$
- If $m \approx n$, about the same
- If $m \gg n$: Householder QR requires about twice as much work as normal equations
- SVD: $mn^2 + n^3$ (with a large constant)

Demo: Relative cost of matrix factorizations [cleared]

In-Class Activity: Householder, Givens, SVD

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