- Feedback
- Exam 2: starts next Friday
- Goals:
    - eigenvalues
        - linalg review, motivation
        - sensitivity
        - methods

Eigenvalue Problems: Setup/Math Recap

A is an $n \times n$ matrix.

- $\boldsymbol{x} \neq 0$ is called an *eigenvector* of A if there exists a λ so that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}. \qquad \iff \qquad A(2\vec{x}) = \lambda(2\vec{x})$$

- In that case, λ is called an *eigenvalue*.
- The set of all eigenvalues $\lambda(A)$ is called the *spectrum*.
- The *spectral radius* is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\}$$

$F = m\,a$ $\qquad\qquad x(t)$

$$a(t) = \frac{\partial^2 x(t)}{\partial t^2}$$

Force by Hooke's law is $k(x_1 - x_2)$

For many springs and many particles,

$$F(t) = A\vec{x}(t)$$

$$Ax(t) = m\frac{\partial^2 x(t)}{\partial t^2} \qquad x(t) = \vec{x}_0 \sin(\omega t)$$

$$A\vec{x}_0 \cancel{\sin(\omega t)} = m\,\vec{x}_0(-\omega^2)\cancel{\sin}(\omega t)$$

$$A\vec{x}_0 = (-\omega^2)\,\vec{x}_0$$

Finding Eigenvalues

How do you find eigenvalues?

$$A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$$
$$\Leftrightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

$\det(A - \lambda I)$ is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for $n \geqslant 5$ is no general formula for roots of polynomial. IOW: no.

- For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *iterate*.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

Multiplicity

$p = (\lambda - 1)^3$

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- *Algebraic Multiplicity*: multiplicity of the root of the characteristic polynomial
- *Geometric Multiplicity*: #of lin. indep. eigenvectors

In general: AM $\geqslant$ GM.
If AM $>$ GM, the matrix is called *defective*.

An Example

defective

Give characteristic polynomial, eigenvalues, eigenvectors of

$$\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}.$$

Char polys: $(1-\lambda)^2 \Rightarrow 1$ with mult 2.

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \iff x + y = x \Rightarrow \underline{y = 0}$$

$$\begin{pmatrix} ? \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

Can't always diagonalize : <u>can</u> always bumpy-diagonalize

Diagonalizability

"Jordan block"

"Jordan normal form"

When is a matrix called *diagonalizable*? if not defective

$\deg(C(P)) = n \overset{AM=GM}{\Rightarrow} \exists n$ lin indep eigenvectors, (across all eigenvalues)

$X = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \leftarrow$ matrix of eigenvectors

$$A\vec{v}_i = \lambda_i \vec{v}_i$$

$$AX = \begin{pmatrix} | & & | \\ \lambda_1 \vec{v}_1 & \cdots & \lambda_n \vec{v}_n \\ | & & | \end{pmatrix} = \cancel{DX} \; XD$$

"similarity transform"

$AX = XD \quad | \cdot X^{-1} \quad \Leftrightarrow \quad X^{-1}AX = D \leftarrow$ "diagonalizing"

$$D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Similar Matrices

$\lambda_n)$

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that $A = XBX^{-1}$.

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose $A\vec{x} = \lambda\vec{x}$ $(\vec{x} \neq \vec{0})$. $B = X^{-1}AX$

Want: $B\vec{y} = \lambda\vec{y}$? $\vec{y} = ?$

Attempt 1: $\vec{y} = X\vec{x}$

$B\vec{y} = X^{-1}AXX\vec{x}$ ...?

Attempt 2: $\vec{y} = X^{-1}\vec{x}$

$B\vec{y} = X^{-1}A\underbrace{XX^{-1}}_{I}\vec{x} = X^{-1}A\vec{x} = X^{-1}\lambda\vec{x}$

$= \lambda\vec{y}$

Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ do?

Shift. $A \to A - \sigma I = B$

$$B\vec{x} = (A - \sigma I)\vec{x} = \lambda\vec{x} - \sigma x = (\lambda - \sigma)\vec{x}$$

Inversion. $A \to A^{-1} = B$

$$A\vec{x} = \lambda\vec{x} \quad | A^{-1} \cdot \iff \frac{\vec{x}}{\lambda} = A^{-1}\vec{x}$$

$$B\vec{x} = A^{-1}\vec{x} \overset{?}{=} \frac{1}{\lambda}\vec{x}$$

Power. $A \to A^k$

$$A^3\vec{x} = AA(\lambda A) = \lambda^3\vec{x} \qquad A^k\vec{x} = \lambda^k\vec{x}$$

Eigenvalue Transformations (II)

Polynomial $A \to aA^2 + bA + cI$ $=B$

$B\vec{x} = aA^2\vec{x} + bA\vec{x} + c\vec{x} = (a\lambda^2 + b\lambda + c)\vec{x}$

Similarity $\underbrace{T^{-1}AT}_{=B}$ with T invertible $A\vec{x} = \lambda\vec{x}$

$B\vec{y} = \lambda\vec{y}$. $\vec{y} = T^{-1}\vec{x}$

Sensitivity (I)

Assume A not defective. Suppose $X^{-1}AX = D$. Perturb $A \to A + E$.
What happens to the eigenvalues?

$$X^{-1}(A+E)X = D+F$$

Want to understand eigenvalues of $A+E$. $A+E$ similar to $D+F$
$\Rightarrow$ same eigenvalues. Idea: understand eigenvalues of $D+F$.

Suppose: $(D+F)\vec{v} = \mu\vec{v}$ $\quad (\vec{v} \neq \vec{0})$

$$F\vec{v} = (\mu I - D)\vec{v} \quad \Big|\, (\mu I - D)^{-1}\cdot$$

Assume $\mu \notin \sigma(A)$. Then $(\mu I - D)$ is invertible.

$$(\mu I - D)^{-1}F v = \vec{v}$$

$$\Rightarrow \cancel{\|\vec{v}\|} \le \|(\mu I - D)^{-1}\| \, \|F\| \, \cancel{\|v\|} \qquad \frac{1}{\|(\mu I - D)^{-1}\|} \le \|F\|$$

Sensitivity (II)

subtract off $X^{-1}AX = D$

$X^{-1}(A+E)X = D + F$. Have $\|(\mu I - D)^{-1}\|^{-1} \le \|F\|$.

Demo: Bauer-Fike Eigenvalue Sensitivity Bound [cleared]

$\to$ from demo: $\dfrac{1}{\|(\mu I - D)^{-1}\|} = |\mu - \lambda_k|$ where λ_k is eigval of A closest to μ.

$X^{-1}EX = F$

$|\mu - \lambda_k| \le \|F\| = \|X^{-1}EX\| \le \operatorname{cond}(X)\,\|E\|$

Power Iteration

Demo: Motivating Power Iteration [cleared]

Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$ with eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.
Further assume $\|\boldsymbol{x}_i\| = 1$.