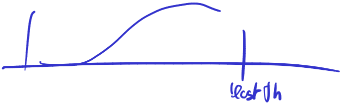
- Homework poll → form
- Final exam available for scheduling → Prairie test
- Exam 2 starts Friday



Goals:

- review power iteration, shift, invert
- multiple eigenvectors
  - Schur factorisation
  - methods → QR iteration

Power Iteration

Demo: Motivating Power Iteration [cleared]

Let $A \in \mathbb{R}^{n \times n}$ and $A\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$ ($j \in \{1, 2, \ldots, n\}$) and $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

Pick some $\boldsymbol{x}_0$, consider $\boldsymbol{x}_{i+1} = A\boldsymbol{x}_i$ ($i \in \{0, \ldots\}$). Called Power Iteration.

Let $\boldsymbol{x}_0 = \sum_{j=1}^{n} \alpha_j \boldsymbol{v}_j$. Observe that $\boldsymbol{x}_i = A^i \boldsymbol{x}_0 = \sum_{j=1}^{n} \alpha_j \lambda^i \boldsymbol{v}_j$.

Define $\boldsymbol{e}_i = \boldsymbol{x}_i / \lambda_1^i - \alpha_1 \boldsymbol{v}_1$.

def

plugging

$$\|\boldsymbol{e}_{i+1}\| = \left\| \frac{\boldsymbol{x}_{i+1}}{\lambda_1^{i+1}} - \alpha_1 \boldsymbol{v}_1 \right\| = \left\| \frac{\sum_{j=1}^{n} \alpha_j \lambda_j^{i+1} \boldsymbol{v}_j}{\lambda_1^{i+1}} - \cancel{\alpha_1 \boldsymbol{v}_1} \right\|$$

$$= \left\| \sum_{j=2}^{n} \alpha_j \left(\frac{\lambda_j}{\lambda_1} \right)^{i+1} \boldsymbol{v}_j \right\| \leqslant \left| \frac{\lambda_2}{\lambda_1} \right|^{i+1} \left\| \sum_{j=2}^{n} \alpha_j \boldsymbol{v}_j \right\| = \left| \frac{\lambda_2}{\lambda_1} \right|^{i+1} \|\boldsymbol{e}_0\|.$$

$\le \left|\frac{\lambda_2}{\lambda_1}\right|^{i+1}$ $\qquad \to 0$

I.e. converges to (a multiple of) $\boldsymbol{v}_1$ 'linearly' (see later).

<1

Power Iteration: Issues?

What could go wrong with Power Iteration?

- Starting vector has no component along $\cancel{z_1}$ $\vec{v}_1$
  Not a problem in practice: Rounding will introduce one.
- Overflow in computing λ_1^i
  $\to$ Normalize after each step
- $|\lambda_1| = |\lambda_2|$
  - If $\lambda_1 = \lambda_2$: multipliclity, defer.
  - If $\lambda_1 \neq \lambda_2$: use shift+invert to separate magnitudes
- Complex eigenvalues
  $\to$ use complex-valued shift, and invert.

Convergence of Power Iteration: Notation

- $\lambda_{\max}(A)$: biggest eigenvalue by magnitude
- $\lambda_{\max 2}(A)$: second-biggest eigenvalue by magnitude.
- $\lambda_{\min 2}(A)$: second-smallest eigenvalue by magnitude
- $\lambda_{\min}(A)$: smallest eigenvalue by magnitude

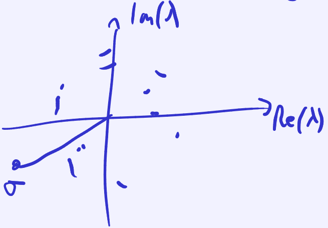
(Not well-defined if there are multiple λ with the same magnitudes. Assume that's not the case.)

Power Iteration: Shift

How does a shift $(A - \sigma I)$ change power iteration?

$\in \mathbb{C} \leftarrow$ shift

$\sigma(A)$ ← spectrum

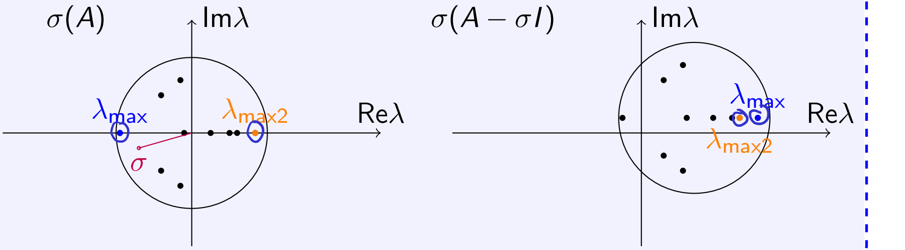


$\sigma(A - \sigma I)$

Power Iteration: Shift

How does a shift $(A - \sigma I)$ change power iteration?

- Converges to eigenvector for $\lambda_{\max}(A - \sigma I)$ with convergence factor $\left|\frac{\lambda_{\max 2}(A-\sigma I)}{\lambda_{\max}(A-\sigma I)}\right|$.
- Can help guide convergence to eigenvalues 'on boundary' of spectrum.



Power Iteration: Inversion

How does inversion (A^{-1}) change power iteration?

$$A\vec{x} = \lambda\vec{x} \quad \Big|\, A^{-1}\cdot \quad \Leftrightarrow \quad \vec{x} = \lambda A^{-1}\vec{x}$$

$$\Leftrightarrow \quad A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

$$\left|\frac{\lambda_{max2}(A^{-1})}{\lambda_{max}(A^{-1})}\right| = \left|\frac{1/\lambda_{min2}(A)}{1/\lambda_{min}(A)}\right| = \left|\frac{\lambda_{min}(A)}{\lambda_{min2}(A)}\right|$$

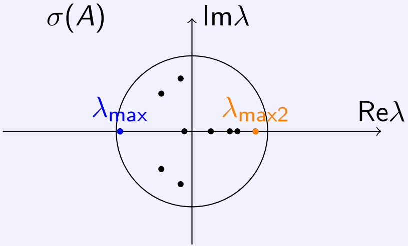
Power Iteration: Inversion

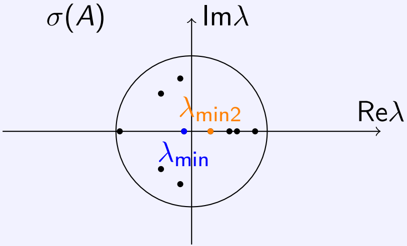
How does inversion (A^{-1}) change power iteration?

- Converges to eigenvector for $\lambda_{\max}(A^{-1}) = 1/\lambda_{\min}(A)$ with convergence factor

$$\left|\frac{\lambda_{\max 2}(A^{-1})}{\lambda_{\max}(A^{-1})}\right| = \left|\frac{1/\lambda_{\min 2}(A)}{1/\lambda_{\min}(A)}\right| = \left|\frac{\lambda_{\min}(A)}{\lambda_{\min 2}(A)}\right|.$$

- Guide convergence to smallest eigenvalues.





Power Iteration: Shift and Inversion

How does shift-invert ($(A - \sigma I)^{-1}$) change power iteration?

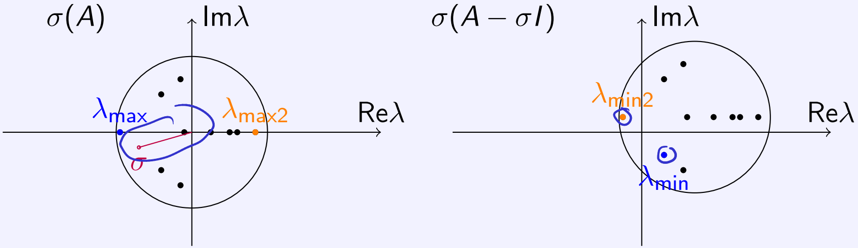
Power Iteration: Shift and Inversion

How does shift-invert ($(A-\sigma I)^{-1}$) change power iteration?

- Converges to eigenvector for $\lambda_{\max}((A-\sigma I)^{-1}) = 1/\lambda_{\min}(A-\sigma I)$ with convergence factor

$$\left|\frac{\lambda_{\max 2}((A-\sigma I)^{-1})}{\lambda_{\max}((A-\sigma I)^{-1})}\right| = \left|\frac{\lambda_{\min}(A-\sigma I)}{\lambda_{\min 2}(A-\sigma I)}\right|.$$

- Guide convergence to eigenvalue closest to σ.



What about Eigenvalues?

Power Iteration generates eigenvectors. What if we would like to know eigenvalues?

Rayleigh quotient $\frac{x^TAx}{x^Tx}$

If we choose RQ as a shift, we get Rayleigh quotient iteration.

$$\frac{\|Ax-\lambda x\|}{|\lambda|\|x\|}$$

Demo: Power Iteration and its Variants [cleared]

Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

$A\vec{v} = \lambda_1 \vec{v}$ $\qquad V = \text{span}\{\vec{v}\}$

$A : V \to V$

$V^\perp \to \mathbb{R}^n = V \oplus V^\perp$ $\qquad V^\perp$

$$A = \underbrace{\begin{bmatrix} \vec{v} & \text{— Basis of } V^\perp \text{—} \end{bmatrix}}_{Q_1} \begin{bmatrix} \lambda & & \\ 0 & & \\ \vdots & ? & \\ 0 & & \end{bmatrix} Q_1^T$$

$Q \; \boxed{\diagdown} \; Q^T$

Schur Form: Comments, Eigenvalues, Eigenvectors

$A = QUQ^T$. For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag.

If we had a Schur form of A (no 2×2 blocks), can we find the eigenvalues?

And the eigenvectors?