- Exam 2
- Vote on hw poll
- Follow-ups:
  - complex starting vec on real valued matrix

    $|a+ib| = |a-ib|$

  - $\vec{x}_j$ $\qquad A\vec{v}_i = \lambda_i \vec{v}_i \qquad V = (\vec{v}_1 \cdots \vec{v}_n)$

    $\vec{x}_j = \sum \alpha_{ji} \vec{v}_i \qquad \vec{\alpha}_j = V^{-1}\vec{x}_j$

symm.
  - $\nabla RQ(\vec{x}) = 0 \qquad \|\vec{x} - \vec{v}_i\| = h \Rightarrow |RQ(x) - \lambda_i| = O(h^2)$ $(h \to 0)$
  - $\nabla RQ(x) = 0 \qquad x$ is eigvec for $\lambda_{min} \to$ min, $\lambda_{max} \to$ max

Goals:

- Schur form
- methods ?
  - deflation
  - orth. it.
  - QR it.
- Krylov

$A = XDX^{-1}$ → doesn't always exist

→ X^{-1} bad if poorly condition

Schur form

Show: Every matrix is orthonormally similar to an upper triangular matrix, i.e. $A = QUQ^T$. This is called the Schur form or Schur factorization.

$A\vec{v} = \lambda\vec{v}$ $\qquad V = \mathrm{span}(\vec{v})$

$A: \quad V \to V$

$\qquad V^\perp \to V \oplus V^\perp$ $\qquad$ (for normal A: $V^\perp$)

$$A = \underbrace{\begin{pmatrix} | & \\ \vec{v} & \text{some orth. basis of } V^\perp \\ | & \end{pmatrix}}_{Q_1} \underbrace{\begin{pmatrix} \lambda & \cdots \\ 0 & \\ \vdots & A_2 \\ 0 & \end{pmatrix}}_{U_1} Q_1^T$$

$Q_1^T A Q_1 = U_1$

$A = Q_1 U_1 Q_1^T$

ONB : $\vec{v}_1 \; \vec{v}_2 \ldots \vec{v}_n$

$$\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots + \alpha_n \vec{v}_n$$

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = V^{-1} \vec{x} = V^T \vec{x}$$

$$V = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{pmatrix}$$

Do that recursively, on A_2

$$A = Q_1 \begin{bmatrix} \lambda & \sim\sim\sim \\ & A_2 \end{bmatrix} Q_1^T \qquad A_2 = Q_2 \begin{bmatrix} \lambda_2 & \\ & \end{bmatrix}$$

. . .

until

$$A = Q \begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix} Q^T \quad \leftarrow \text{Schur form.}$$

eigenvalues on diagonal

Schur Form: Comments, Eigenvalues, Eigenvectors

FYI $A = QUQ^T$. For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag. → with real-valued Schur form

If we had a Schur form of A (no 2×2 blocks), can we find the eigenvalues?

diag. of ∇

And the eigenvectors?

$$U - \lambda_k I = \begin{array}{cc} \overbrace{\quad}^{k-1} & \overbrace{\quad}^{(n-k)} \\ \multicolumn{2}{c}{\begin{pmatrix} U_{11} & \vec{u} & U_{13} \\ & 0 & \vec{w}^T \\ & & U_{33} \end{pmatrix}} \end{array}$$ has a nullspace

$U - \lambda_k I \in \mathbb{R}^{n \times n}$

$$\vec{x} = [U_{11}^{-1}\vec{u}; -1; 0]^T$$

$$\Rightarrow (U - \lambda I)\vec{x} = \vec{0}$$

$$\begin{pmatrix} u^{11} & \vec{u} \\ \vec{0} & \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & \vec{u} & u_{13} \\ & 0 & \vec{w}^T \\ & & u_{33} \end{pmatrix} \begin{pmatrix} u_{11} u_{11}^{-1} \vec{u} - \vec{u} = \vec{0} \\ 0 \\ \vec{0} \end{pmatrix}$$

$$U\vec{x} = \lambda\vec{x} \qquad A = QUQ^T$$

$$\vec{y} = Q\vec{x}$$

$$A\vec{y} = QUQ^T\vec{y} = \lambda\vec{y}$$

computable at $\mathcal{O}(n^2)$ (ops) via back sub

Schur Form: Comments, Eigenvalues, Eigenvectors

$A = QUQ^T$. For complex λ:

- Either complex matrices, or
- 2×2 blocks on diag.

If we had a Schur form of A (no 2×2 blocks), can we find the eigenvalues?

And the eigenvectors?

Computing Multiple Eigenvalues

All Power Iteration Methods compute one eigenvalue at a time.
What if I want *all* eigenvalues?

- Follow argument for Schur form:
  Find SF by reducing the problem size one vector at a time.
  "deflation"
- Power iteration w/ multiple vectors

Simultaneous Iteration

What happens if we carry out power iteration on multiple vectors simultaneously?

$X_0 =$ something random

$X_{i+1} := A X_i$

bad: • all columns conv. to leading

• unnormalized

Orthogonal Iteration

$X_0 =$ something random

$\tilde{X}_{i+1} = A X_i$

$Q_0 R_0 = \tilde{X}_{i+1}$

$X_1 = Q_0$

Toward the QR Algorithm

$$Q_0 R_0 = X_0$$

$$X_1 = A Q_0$$

$$Q_1 R_1 = X_1 = A Q_0 \Rightarrow \quad Q_1 R_1 Q_0^T = A$$

$$\hat{X}_2 = A Q_1$$

$$Q_2 R_2 = X_2$$

If Q_k converge, ... so that $Q_k \approx Q_{k+1}$: $R_k \approx Q_k^T A Q_k = \hat{X}_k$

check $\hat{X}_k$ for "upper-triangular-ness" to see if we've converged to Schur form

Demo: Orthogonal Iteration [cleared]

QR Iteration/QR Algorithm

$X_0 = A$

$Q_k R_k = X_k$

$X_{k+1} = A Q_k$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k = \bar{Q}_k^T \bar{X}_k \bar{Q}_k = \bar{Q}_k^T \bar{Q}_{k-1}^T \cdots \bar{Q}_0^T A \bar{Q}_0 \cdots \bar{Q}_k$

↳ orth similarity transform of A

↳ have same eigenvalues as A.

$\hat{X}_k = \bar{X}_{k+1}$

$\bar{X}_0 = A$

$\bar{Q}_k \bar{R}_k = \bar{X}_0$

$\bar{X}_{k+1} = \bar{R}_k \bar{Q}_k$

Proof sketch: Equivalence of QR iteration/Orth. iteration

Orthogonal Iteration (no bars)

- $X_0 := A$
  - $Q_0 R_0 := X_0$,
  - where we may choose $Q_0 = \bar{Q}_0$
  - $\hat{X}_0 = Q_0^H A Q_0 = Q_0^H Q_0 R_0 Q_0 = R_0 Q_0$
- $X_1 := A Q_0$
  - $Q_1 R_1 := X_1$,
    and because of
    $X_1 = Q_0 Q_0^H A Q_0 = Q_0 \bar{X}_1 = Q_0 \bar{Q}_1 \bar{R}_1$
    we may choose
    $Q_1 = Q_0 \bar{Q}_1 = \bar{Q}_0 \bar{Q}_1$.
- $\vdots$

QR Iteration (with bars)

- $\bar{X}_0 := A$
  - $\bar{Q}_0 \bar{R}_0 := A$
- $\bar{X}_1 := \bar{R}_0 \bar{Q}_0 = \hat{X}_0$
  - $\bar{Q}_1 \bar{R}_1 := \bar{X}_1$
- $\bar{X}_2 := \bar{R}_1 \bar{Q}_1$
  - $\bar{X}_2 = Q_1^H A Q_1 = \hat{X}_1$
- $\vdots$

$$A = Q \qquad Q^T$$