December 3, 2024

Announcements

- Exam 4 grades out
- HW 10 tomorrow

Goals

- quad review
- diff
- diff eq

Review

$$\int_a^b f(x)\,dx \approx \int_a^b p_{n-1}(x) = \sum_{i=1}^{n} f(x_i)\, w_i$$

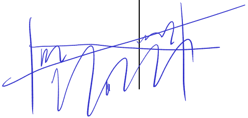
$$f(x_i) = p_{n-1}(x_i)$$

quad. rules

$$E(h) = O(h^p)$$

as $h \to 0$

"order of acc"



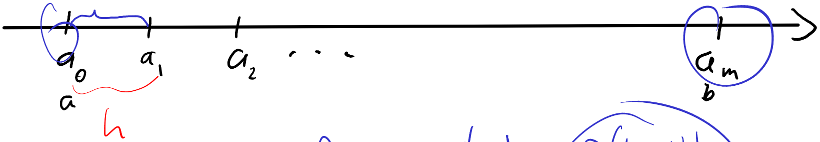
h - refinement

p - refinement

Composite Quadrature

High-order polynomial interpolation requires a high degree of smoothness of the function.
Idea: Stitch together multiple lower-order quadrature rules to alleviate smoothness requirement.

e.g. trapezoidal



$$\text{Quad-E}(h) = O(h^{n+1})$$

Error in Composite Quadrature

What can we say about the error in the case of composite quadrature?

intervals

h^{n+1}

$$\left|\int_a^b f(x)\,dx - \sum_{j=1}^{m} \sum_{i=1}^{n} w_{j,i}\, f(x_{j,i})\right| \le C\, \|f^{(n)}\|_\infty \sum_{j=1}^{m} (a_j - a_{j-1})^{n+1}$$

Fixed at the class

$$= C\, \|f^{(n)}\|_\infty \sum_{j=1}^{m} \underbrace{(a_j - a_{j-1})^{n}}_{\le h^n} (a_j - a_{j-1})$$

$$\le C\, \|f^{(n)}\|_\infty\, h^n \underbrace{\sum_{j=1}^{m} (a_j - a_{j-1})}_{b-a}$$

Composite Quadrature: Notes

Observation: Composite quadrature loses an order compared to non-composite.

Idea: If we can estimate errors on each subinterval, we can shrink (e.g. by splitting in half) only those contributing the most to the error. (adaptivity)

Taking Derivatives Numerically

$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ $\quad (e^{i\alpha x})' = i\alpha e^{i\alpha x}$

Why *shouldn't* you take derivatives numerically?

$1 = \|e^{i\alpha x}\|_\infty \qquad |\alpha| \underbrace{\|e^{i\alpha x}\|_\infty}_{1}$

- ∂_x : unbounded operator $\rightsquigarrow$ conditioning
- cancellation
- amplifies noise

Numerical Differentiation: How?

How can we take derivatives numerically?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x_i) = p_{n-1}(x_i)$$

$$p_{n-1}(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x) \qquad x^0,\ x^1,\ x^2 \qquad \Leftrightarrow \qquad V\vec{\alpha} = p_{n-1}(\vec{x})$$

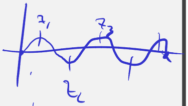
$$p_{n-1}'(x) = \sum_{i=1}^{n} \alpha_i \varphi_i'(x) \qquad \Leftrightarrow \qquad V'\vec{\alpha} = p_{n-1}'(\vec{x})$$

Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Basics)

Numerical Differentiation: Accuracy

$$w(x)\left(\prod_{i=1}^{n}(x-x_i)\right)$$

$$f(x)-p_{n-1}(x)=\frac{f^{(n)}(\xi)}{n!}\overbrace{\prod_{i=1}^{n}(x-x_i)}^{\le h^n}$$



$$f'(x)-p'_{n-1}(x)=\frac{f^{(n)}(\cancel{\xi(x)})}{n!}\left(\prod_{i=1}^{n}(x-x_i)\right)'$$

no! (assume)

$$w'(z_i)=0$$

$$C\cdot\underbrace{\prod_{i=1}^{n-1}(x-z_i)}_{\le h^{n-1}}$$

$$=C\cdot h^{n-1}$$

Demo: Taking Derivatives with Vandermonde Matrices [cleared]

Differentiation Matrices

How can numerical differentiation be cast as a matrix-vector operation?

$$V\vec{\alpha} = p_{n-1}(\vec{x}) = f(\vec{x}) \iff \vec{\alpha} = V^{-1} f(\vec{x})$$

$$V'\vec{\alpha} = p_{n-1}'(\vec{x}) \approx f'(\vec{x}) \iff V'\vec{\alpha} \approx f'(\vec{x})$$

$$f'(\vec{x}) = \underbrace{V' V^{-1}}_{D} f(\vec{x})$$

Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Build D)

How about a derivative at a single node?

$$\begin{pmatrix} f'(x_1) \\ \vdots \\ f'(x_n) \end{pmatrix} \approx D \cdot f(\vec{x}) = \begin{pmatrix} D_1 \cdot f(\vec{x}) \\ \vdots \\ D_n \cdot f(\vec{x}) \end{pmatrix}$$

first row

$$\downarrow$$

$$= \begin{pmatrix} \boxed{} \\ D \\ \end{pmatrix} \begin{pmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{pmatrix}$$

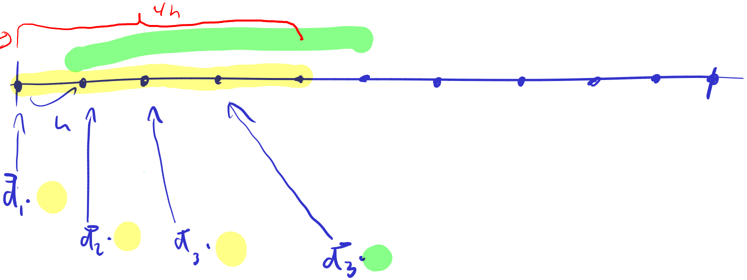
scale: $x_i \mapsto \alpha \cdot x_i$

$D \mapsto D/\alpha$

shift: $x_i \mapsto x_i + \sigma$

$D \mapsto D$

added after class



D_{01} ← diff. with 5 equispace'd nodes on $[0,1]$

$$D_{\cancel{5h}\,4h} = D_{01} / \cancel{5h}\,4h = \begin{pmatrix} \vec d_1^T \\ \vec d_2 \\ \vdots \\ \vec d_5^T \end{pmatrix}$$

fixed after class

Properties of Differentiation Matrices

How do I find second derivatives?

D^2

Does D have a nullspace?

Demo: Taking Derivatives with Vandermonde Matrices [cleared] (Shifting and scaling the nodes)

Numerical Differentiation: Shift and Scale

Does D change if we shift the nodes $(x_i)_{i=1}^n \to (x_i + c)_{i=1}^n$?

Does D change if we scale the nodes $(x_i)_{i=1}^n \to (\alpha x_i)_{i=1}^n$?

Finite Difference Formulas from Diff. Matrices

How do the rows of a differentiation matrix relate to FD formulas?

Assume a large equispaced grid and 3 nodes w/same spacing. How to use?

Finite Differences: via Taylor

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

$$f(x+h) = f(x) + f'(x) \cdot h + f''(x) \cdot \frac{h^2}{2} + \dots$$

$$f^{(15)}(x) = \alpha \cdot f(x+17h) \cdot \dots -$$

More Finite Difference Rules

Similarly:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

(Centered differences)

Can also take higher order derivatives:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

Can find these by trying to match Taylor terms.
Alternative: Use linear algebra with interpolate-then-differentiate to find FD formulas.

Demo: Finite Differences vs Noise [cleared]

Demo: Floating point vs Finite Differences [cleared]

Richardson Extrapolation

Deriving high-order methods is hard work. Can I just do multiple low-order approximations (with different h and get a high-order one out?

Suppose we have $F = \tilde{F}(h) + O(h^p)$ and $\tilde{F}(h_1)$ *and* $\tilde{F}(h_2)$.