December 5, 2024

Announcements

- Final
- HW10
- ICES
- Bring questions for Thu.

Goals

- Calculus cheat sheet for the final
- Richardson
- IVPs

Review

- CS555: Numerical Methods for PDE
- CS556: O(n) ~~mmm~~ linear algebra
- CS554: Parallel numerics

CS598 EVS: tensor

APK integral equations

Richardson Extrapolation

Deriving high-order methods is hard work. Can I just do multiple low-order approximations (with different h and get a high-order one out?

Suppose we have $F = \tilde{F}(h) + O(h^p)$ and $\tilde{F}(h_1)$ *and* $\tilde{F}(h_2)$.

Assume: $F = \tilde{F}(h) + a h^p + O(h^q)$ often $q = p+1$

not known

$F = \alpha \tilde{F}(h_1) + \beta \tilde{F}(h_2) + O(h^q)$

$\alpha \cancel{a} h_1^p + \beta \cancel{a} h_2^p = 0$, $\alpha + \beta = 1$

$\alpha h_1^p + (1-\alpha) h_2^p = 0$ $\Leftrightarrow$ $\beta = 1 - \alpha$

$\alpha (h_1^p - h_2^p) + h_2^p = 0$ $\Leftrightarrow$ $\alpha = \dfrac{-h_2^p}{h_1^p - h_2^p}$

Richardson Extrapolation: Observations,

What are α and β for a first-order (e.g. finite-difference) method if we choose $h_2 = h_1/2$?

$$\alpha = \frac{-h_2^p}{h_1^p - h_2^p} = \frac{-\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = -1 \qquad \beta = 2.$$

Demo: Richardson with Finite Differences [cleared]

Romberg Integration

Can this be used to get *even higher order* accuracy?

h

$h/2$

$h/4$

$h/8$

Outline

What can we solve already?

- Linear Systems: yes
- Nonlinear systems: yes
- Systems with derivatives: no

Some Applications

$y_1'(t) = \alpha y_1(t)$

IVPs	BVPs
▶ Population dynamics $y_1' = y_1(\alpha_1 - \beta_1 y_2)$ (prey) $y_2' = y_2(-\alpha_2 + \beta_2 y_1)$ (predator) ▶ chemical reactions ▶ equations of motion	▶ bridge load ▶ pollutant concentration (steady state) ▶ temperature (steady state) ▶ waves (time-harmonic)

Demo: Predator-Prey System [cleared]

$\vec{y}_0 = \vec{y}(0)$

$y'(t) = f(\vec{y}(t))$

$\vec{y}(h) = \vec{y}_0 + f(\vec{y}_0) \cdot h$

Initial Value Problems: Problem Statement

Want: Function $\boldsymbol{y} : [0, T] \to \mathbb{R}^n$ so that

- $\boldsymbol{y}^{(k)}(t) = \boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k-1)})$ (*explicit*), or
- $\boldsymbol{f}(t, \boldsymbol{y}, \boldsymbol{y}', \boldsymbol{y}'', \ldots, \boldsymbol{y}^{(k)}) = 0$ (*implicit*)

are called explicit/implicit *kth-order ordinary differential equations* (*ODEs*).
Give a simple example.

Not uniquely solvable on its own. What else is needed?

need initial conditions

$y(0) = y_0 \quad \ldots \quad y^{(k-1)}(0) = y_{k-1}$

Reducing ODEs to First-Order Form

A kth order ODE can always be reduced to first order. Do this in this example:

$$y''(t) = f(y)$$

$y' = v$

$v' = w$

$w' = \ldots$

$u = y$

$v = y'$

$u' = v$

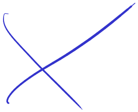
$v' = (y'') = f(y)$

Conditioning

Unfortunate terminology accident: "Stability" in ODE-speak

To adapt to conventional terminology, we will use 'Stability' for

- the conditioning of the IVP, *and*
- the stability of the methods we cook up.



Some terminology:

An IVP is stable if and only if...

An IVP is asymptotically stable if and only if

Example I: Scalar, Constant-Coefficient

$$\begin{cases} y'(t) = \lambda y \\ y(0) = y_0 \end{cases} \quad \text{where} \quad \lambda = a + ib$$

Solution?

When is this stable?

Example II: Constant-Coefficient System

$$\begin{cases} \boldsymbol{y}'(t) = A\boldsymbol{y}(t) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$

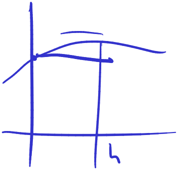
Assume $V^{-1}AV = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ diagonal. Find a solution.

When is this stable?

Euler's Method

Discretize the IVP

$$\begin{cases} \boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}) \\ \boldsymbol{y}(t_0) = \boldsymbol{y}_0 \end{cases}$$



- Discrete times: $t_1, t_2, \ldots,$ with $t_{i+1} = t_i + h$
- Discrete function values: $\boldsymbol{y}_k \approx \boldsymbol{y}(t_k)$.

$$\vec{y}(t) = \vec{y}_0 + \int_0^t y' \, d\tau = \vec{y}_0 + \int_0^t f(y(\tau)) \, d\tau$$

$$= \vec{y}_0 + \sum_{i=1}^{n} f(\underbrace{y(\tau_i)}_{\text{not known}}) \, \omega_i$$

left rect. rule

$$\approx \vec{y}_0 + f(y_0)\, h$$

not known

right rect. rule

backward $\longrightarrow$ $\underbrace{y(h)}_{y_{n+1}} = \vec{y}_0 + f(\underbrace{\vec{y}(h)}_{y_{n+1}}) \cdot h$

Euler's method: Forward and Backward

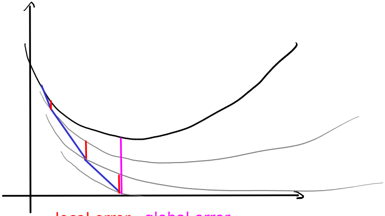
$$\boldsymbol{y}(t) = \boldsymbol{y}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(\tau))\mathrm{d}\tau,$$

Use 'left rectangle rule' on integral:

Use 'right rectangle rule' on integral:

Demo: Forward Euler stability [cleared]

Global and Local Error



Let $u_k(t)$ be the function that solves the ODE with the initial condition $u_k(t_k) = y_k$. Define the local error at step k as...

Define the global error at step k as...

About Local and Global Error

Is global error $= \sum$local errors?

A time integrator is said to be *accurate of order p* if...

ODE IVP Solvers: Order of Accuracy

A time integrator is said to be *accurate of order p* if $\ell_k = O(h^{p+1})$

This requirement is one order higher than one might expect–why?

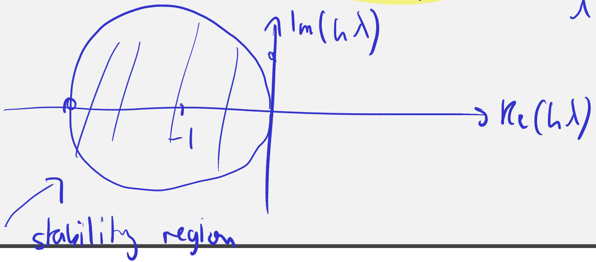
Stability of a Method

Find out when forward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_n = y_{n-1} + h\lambda y_{n-1}$$
$$= y_{n-1}(1+h\lambda)$$
$$= y_0(1+h\lambda)^k$$

stable $\iff$ $|1+h\lambda| < 1$

$\lambda \in \mathbb{C}$



Stability: Systems

What about stability for systems, i.e.

$$\mathbf{y}'(t) = A\mathbf{y}(t)?$$

diagonalize

Stability: Nonlinear ODEs

What about stability for nonlinear systems, i.e.

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t))?$$

linearize, diagonalize.

Stability for Backward Euler

Find out when backward Euler is stable when applied to $y'(t) = \lambda y(t)$.

$$y_n = y_{n-1} + h\lambda y_n$$

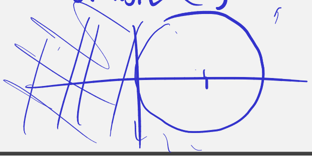
$$y_n(1-h\lambda) = y_{n-1}$$

$$y_n = y_{n-1}\left(\frac{1}{1-h\lambda}\right) \quad |\cdot|<1$$

$$= y_0\left(\frac{1}{1-h\lambda}\right)^k$$

n/A

stable $\Leftrightarrow$ $|1-h\lambda| \geq 1$



ODE stable $\Leftrightarrow$ Re $\lambda < 0$

all stable w/ BwEuler

Demo: Backward Euler stability [cleared]