HW1

Broadcasting:

rows colums

A: (4, 4)

x: (1 (4,)

x	y
$(2, n_x)$	$(2, n_y)$
↳ $(2, n_x, ?)$	↳ $(2, ?, n_y)$
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$	

all dist vectors have shape:

$(2, n_x, n_y)$

Recap: Norms

What's a norm?

Define *norm*.

- Definiteness
- scaliy
- tringle ineq

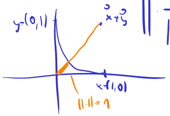
Norms: Examples

Examples of norms?

$\vec{x} \in \mathbb{R}^n$ $\qquad \|\vec{x}\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} \qquad (p \geq 1)$

$p = \infty \qquad \|\vec{x}\|_\infty = \max_{i=1}^{n} |x_i|$

Demo: Vector Norms [cleared]



$\left\| \frac{x}{\|x\|} \right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$

$\underbrace{\|\vec{x}+\vec{y}\|}_{>2} \leq \overset{1}{\|x\|} + \overset{1}{\|y\|} = 2$

Norms: Which one?

Does the choice of norm really matter much?

Suppose you have $\|\cdot\|$, $\|\cdot\|_*$.
Infinite-dim, there exist $\alpha, \beta > 0$ so that

$$\alpha \|\vec{x}\| \le \|\vec{x}\|_* \le \beta \|\vec{x}\| \qquad (\vec{x} \in \mathbb{R}^n)$$

In these notes: If we write $\|\cdot\|$ without any specifics, then the statement is true for any norm. If a specific norm is needed, the notation will indicate that.

Norms and Errors

If we're computing a vector result, the error is a vector.
That's not a very useful answer to 'how big is the error'.
What can we do?

$\vec{x}$: true
$\vec{y}$: computed

$$\cancel{\frac{\|\vec{x}\| - \|\vec{y}\|}{\|\vec{x}\|}}$$

correct:

$$\frac{\|\vec{x} - \vec{y}\|}{\|\vec{x}\|}$$

Forward/Backward Error

Suppose *want* to compute $y = f(x)$, but *approximate* $\hat{y} = \hat{f}(x)$.

What are the forward error and the backward error?

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, dash] \arrow[dr, "\hat{f}"] & y \arrow[d, dashed, dash] \\ \hat{x} \arrow[r, "f"'] & \hat{y} \end{tikzcd}$$

$\overbrace{|y - \hat{y}|}^{\Delta y}$: forward error

$\overbrace{|x - \hat{x}|}^{\Delta x}$: backward error

Forward/Backward Error: Example

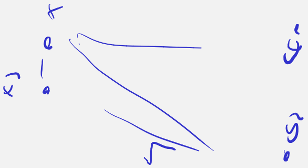
Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the forward error?

rel. fw. error $= \dfrac{|\sqrt{2} - 1.4|}{\sqrt{2}} \approx 0.01\ldots = 1.\ldots \cdot 10^{-2}$

"accurate to two sig. digits"

Forward/Backward Error: Example

Suppose you wanted $y = \sqrt{2}$ and got $\hat{y} = 1.4$.
What's the (magnitude of) the backward error?



$f(\hat{x}) = \hat{y} \iff \sqrt{\hat{x}} = 1.4 \qquad \hat{x} = 1.4^2$

$\hat{x} = 1.96 \qquad \frac{|\hat{x} - x|}{|x|} \approx 0.02$

Forward/Backward Error: Observations

What do you observe about the relative magnitude of the relative errors?

Forward/Backward Error: Observations

What do you observe about the relative magnitude of the relative errors?

- In this case: Got smaller, i.e. variation *damped out*.
- Typically: Not that lucky: Input error *amplified*.
- If backward error is smaller than the input error:
  result "as good as possible".

This amplification factor seems worth studying in more detail.

Sensitivity and Conditioning

Consider a more general setting: An input x and its perturbation $\hat{x}$.

$$\frac{|f(x) - f(\hat{x})|}{|f(x)|} \leq \kappa_{rel} \cdot \frac{|x - \hat{x}|}{|x|}$$

(relative) condition number

$$\kappa_{rel} = \max_{x, \hat{x}} \frac{\dfrac{|f(x) - f(\hat{x})|}{|f(x)|}}{\dfrac{|x - \hat{x}|}{|x|}}$$

Absolute Condition Number

Can you also define an *absolute* condition number?

$$\kappa_{abs} = \max_{x,\hat{x}} \frac{|f(x) - f(\hat{x})|}{|x - \hat{x}|}$$

Absolute Condition Number

Can you also define an *absolute* condition number?

Certainly:

$$\kappa_{\text{abs}} = \max_{x,\hat{x}} \frac{|f(x) - f(\hat{x})|}{|x - \hat{x}|}$$

But: less commonly used than relative, because we *typically* care about relative error.

When not specified: Assume condition number means *relative*.

Interpreting a Condition Number

What does it mean for condition numbers to be small/large?

small: insensitive

large: sensitive

Relate the (relative) condition number back to the setting of (relative) backward error.