#### October 10, 2024 Announcements

- Exam - HW 5

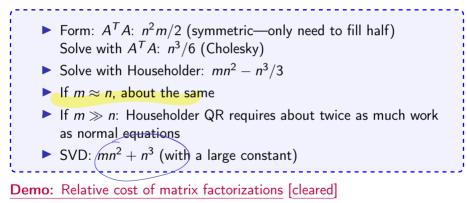
#### Goals

- Elgenvalues - CA tevient - Sensitivity - Mohods

#### Review

# Comparing the Methods

Methods to solve least squares with A an  $m \times n$  matrix:



# Outline

Introduction to Scientific Computing

Systems of Linear Equations

Linear Least Squares

Eigenvalue Problems Properties and Transformations Sensitivity Computing Eigenvalues Krylov Space Methods

Nonlinear Equations

Optimization

Interpolation

Numerical Integration and Differentiation

Initial Value Problems for ODEs

Boundary Value Problems for ODEs

Partial Differential Equations and Sparse Linear Algebra

Fast Fourier Transform

Additional Topics

Eigenvalue Problems: Setup/Math Recap

A is an  $n \times n$  matrix.

•  $x \neq 0$  is called an *eigenvector* of A if there exists a  $\lambda$  so that

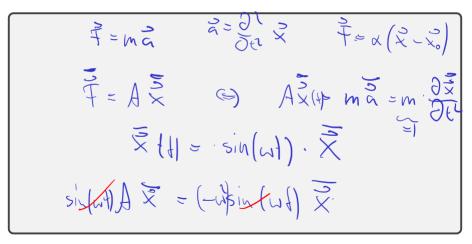
#### $A\mathbf{x} = \lambda \mathbf{x}.$

- ln that case,  $\lambda$  is called an eigenvalue.
- The set of all eigenvalues  $\lambda(A)$  is called the *spectrum*.
- ▶ The *spectral radius* is the magnitude of the biggest eigenvalue:

$$\rho(A) = \max\{|\lambda| : \lambda(A)\} \quad O(A)$$

### Eigenvalue Problems: Motivation from Mechanics

Consider mass-spring systems, e.g. as modeled in (e.g.) <u>myphysicslab.com</u> What is needed to model?



# Finding Eigenvalues

How do you find eigenvalues?

$$A\mathbf{x} = \lambda \mathbf{x} \Leftrightarrow (A - \lambda I)\mathbf{x} = 0$$
  
$$\Rightarrow A - \lambda I \text{ singular} \Leftrightarrow \det(A - \lambda I) = 0$$

,x+0

det $(A - \lambda I)$  is called the *characteristic polynomial*, which has degree n, and therefore n (potentially complex) roots.

Does that help algorithmically? Abel-Ruffini theorem: for  $n \ge 5$  is no general formula for roots of polynomial. IOW: no.

- ▶ For LU and QR, we obtain *exact* answers (except rounding).
- For eigenvalue problems: not possible—must *iterate*.

Demo: Rounding in characteristic polynomial using SymPy [cleared]

# Multiplicity

What is the *multiplicity* of an eigenvalue?

Actually, there are two notions called multiplicity:

- Algebraic Multiplicity: multiplicity of the root of the characteristic polynomial
- ► Geometric Multiplicity: #of lin. indep. eigenvectors

In general:  $AM \ge GM$ .

If AM > GM, the matrix is called *defective*.

## An Example

Give characteristic polynomial, eigenvalues, eigenvectors of

$$(P: (I-\lambda)^{2})$$

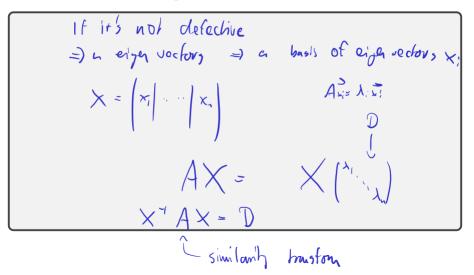
$$AM: Z For both eigenvalues of I$$

$$A(\frac{x}{y}) = (\frac{x}{y}) \in y + y = x \in y_{0}$$

 $\begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ .

# Diagonalizability

When is a matrix called *diagonalizable*?



## Similar Matrices

Related definition: Two matrices A and B are called similar if there exists an invertible matrix X so that  $A = XBX^{-1}$ .

 $X^{'}A = \beta \chi^{'}$  $X^{'}A \chi = \beta$ 

In that sense: "Diagonalizable" = "Similar to a diagonal matrix".

Observe: Similar A and B have same eigenvalues. (Why?)

Suppose 
$$A \overset{\sim}{\mathcal{C}} = \lambda \overset{\sim}{\mathcal{C}} (\overset{\sim}{\mathcal{X}} \neq 0)$$
.  
 $B \overset{\sim}{\mathcal{U}} = X^{-1} A \overset{\sim}{\mathcal{X}} \times X^{-1} \overset{\sim}{\mathcal{C}} = X^{-1} A \overset{\sim}{\mathcal{U}} = \lambda \overset{\sim}{\mathcal{U}} = \lambda \overset{\sim}{\mathcal{U}}$ 

# Eigenvalue Transformations (I)

What do the following transformations of the eigenvalue problem  $A\mathbf{x} = \lambda \mathbf{x}$ do? Shift.  $A \to A - \sigma I$ 

$$(A - \delta I) \vec{v} = A \vec{v} - \sigma \vec{v} = \lambda \vec{v} - \sigma$$

Inversion.  $A \rightarrow A^{-1}$ 

$$A \overrightarrow{x} = \lambda \overrightarrow{x} | A' \cdot (-) \overrightarrow{x} = \lambda A' \overrightarrow{x} - A' \overrightarrow{x} = A' \overrightarrow{x}$$

Power.  $A \to A^k$   $A^3 = A A A$ 

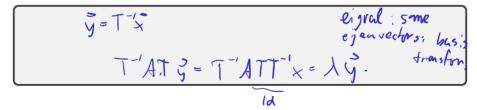
$$A^{k} \stackrel{\sim}{\times} = A^{k+1} A_{\times} = A^{k+1} \lambda_{\times}^{*} = \dots = \lambda^{k} \stackrel{\sim}{\times}$$

# Eigenvalue Transformations (II)

Polynomial 
$$A \rightarrow aA^2 + bA + cI$$

$$(aA^{2}+bA+cJ) \stackrel{>}{\times} = (a\lambda^{2}+b\lambda+c) \stackrel{>}{\times}$$

Similarity  $T^{-1}AT$  with T invertible



# Sensitivity (I)

Assume A not defective. Suppose  $X^{-1}AX = D$ . Perturb  $A \rightarrow A + E$ . What happens to the eigenvalues?

$$\begin{array}{c} X^{-1}(A+E) \times = D + \mp \quad (\mp not accessing) \\ Suppose \vec{v} \neq \vec{0} \text{ is eigenvector of } D + \mp_{1} \quad (\exists not accessing) \\ (D+\mp) = \mu^{-1} \\ \mp\vec{v} = (\mu I - D) \vec{v} \quad (\mu I - D)^{-1} \quad (\mu I - D)^{-1} \quad of A. \end{array}$$

Sensitivity (II)  $X^{-1}(A+E)X = D + F$ . Have  $\|(\mu I - D)^{-1}\|^{-1} \le \|F\|$ . Demo: Bauer-Fike Eigenvalue Sensitivity Bound [cleared]

$$\| (\mu T - D)^{-1} \|^{-1}$$
 is the distance between  
 $\mu$  and the closest eigenvalue  
of A top  
 $\| \mu - \lambda_{k} \| \leq \| \| \mp \| = \| \times^{-1} \in \chi(|x|) \| \in \|$   
 $\mp = \chi^{-1} \in \chi$