CS 450: Numerical Anlaysis

Lecture 4

Chapter 2 – Linear Systems

Orthogonal Matrices and Conditioning of Linear Systems

Edgar Solomonik

Department of Computer Science
University of Illinois at Urbana-Champaign

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Norms and Conditioning of Orthogonal Matrices

- **Orthogonal matrices:** $Q^{-1} = Q^T$

$$\|Qx\|_2 = \|x\|_2$$

$$\|QA\|_F = \|A\|_F$$

- **Norm and condition number of orthogonal matrices:**

$$\|Q\|_2 = 1$$

$$= \max_{\|x\|_2 = 1} (\langle Qx, Qx \rangle)^{1/2} = \max_{\|x\|_2 = 1} (x^T \underbrace{Q^T Q}_{I} x)^{1/2}$$

$$= \max_{\|x\|_2 = 1} (x^T x)^{1/2} = 1$$

$$\kappa(A) = \|Q\|_2 \cdot \|Q^{-1}\|_2$$

$$= 1 \cdot \|Q^T\|_2 = 1$$

Singular Value Decomposition

- **The singular value decomposition (SVD):**

diagonal

nonnegative

decreasing

$$A = U \Sigma V^T$$

orthogonal

$=$

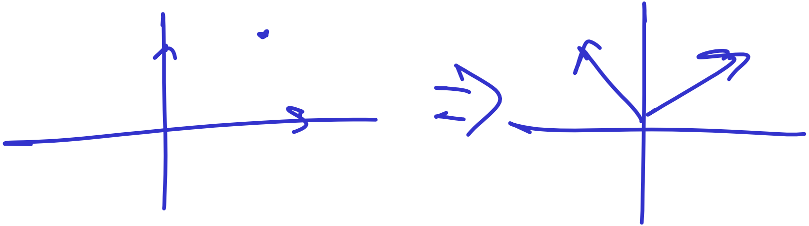
orthogonal

$$\Sigma = \begin{bmatrix} \sigma_{max} & & \\ & \ddots & \\ & & \sigma_{min} \end{bmatrix}$$

U, V^T have columns that are left/right singular vectors

Singular Value Decomposition

- **The singular value decomposition (SVD):**



Qv expresses v in the basis of rows of Q

SVD – a matrix is a diagonal map from vectors in a basis of V to vectors in a basis of U

Singular Value Decomposition

- **The singular value decomposition (SVD):**

$$\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{bmatrix} = V^T x \qquad x = \sum_i \alpha_i v_i$$

$$V^T = \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T\end{bmatrix}$$

Norms and Conditioning using the SVD

- **Norm and condition number in terms of singular values:**

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = V^T x$$

$$\|A\|_2 = \|U \Sigma V^T\|_2$$

$$\le \underbrace{\|U\|_2}_{1} \|\Sigma\|_2 \underbrace{\|V^T\|_2}_{1}$$

$$\le \|\Sigma\|_2 = \sigma_{max}$$

$$\|A\|_2 = \sigma_{max}$$

$$y = U \Sigma \underbrace{V^T x}_{\text{does not change } \|x\|}$$

$$A = U \Sigma V^T$$

$$U^T A V = \Sigma$$

$$\underbrace{\|\Sigma\|_2}_{\sigma_{max}} \le \underbrace{\|U^T\|_2}_{1} \|A\|_2 \underbrace{\|V\|_2}_{1}$$

Norms and Conditioning using the SVD

- **Norm and condition number in terms of singular values:**

minimum growth in norm of x when Ax

$$U \begin{pmatrix} \sigma_{max} & & \\ & \ddots & \\ & & \sigma_{min} \end{pmatrix} \underbrace{V^T x}_{\text{retains } \|x\|_2 = \|V^T x\|_2}$$

$$A^{-1} = (U \Sigma V^T)^{-1}$$

$$= V \Sigma^{-1} U^T$$

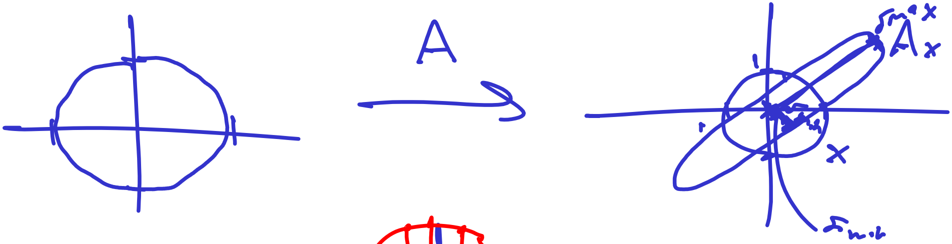
$$V^T x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

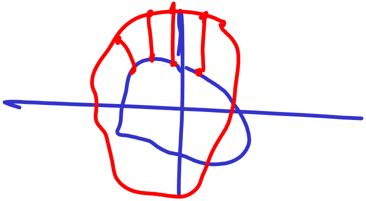
$$\|A^{-1}\|_2 = 1/\sigma_{min}$$

$$\kappa(A) = \frac{\sigma_{max}}{\sigma_{min}}$$

Norms and Conditioning using the SVD

- **Norm and condition number in terms of singular values**:





Conditioning of Linear Systems

- **Lets now return to formally deriving the conditioning of solving $Ax = b$:**

$\hat{b} = b + \delta b$

$A\underbrace{\hat{x}}_{x + \delta x} = \hat{b}$

$A(x + \delta x) = b + \delta b$

$A\,\delta x = \delta b$

would like to bound $\frac{\|\delta x\|}{\|x\|}$ with respect to $\frac{\|\delta b\|}{\|b\|}$

$\delta x = A^{-1}\delta b$

$\|\delta x\| = \|A^{-1}\delta b\|$

$\leq \|A^{-1}\|\,\|\delta b\|$

$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\,\|\delta b\|}{\|x\|}$

$\|x\| \geq \frac{\|b\|}{\sigma_{max}}$

$\leq \kappa(A)\frac{\|\delta b\|}{\|b\|}$

Conditioning of Linear Systems

- **Lets now return to formally deriving the conditioning of solving $Ax = b$:**

$$\frac{\|\delta x\|}{\|x\|} \le \kappa(A) \frac{\|\delta b\|}{\|b\|} = \frac{\sigma_{max} \to \|\delta b\|}{\sigma_{min} \to \|b\|}$$

Conditioning of Linear Systems II

- **Consider perturbations to the input coefficients $\hat{A} = A + \delta A$:**

$$\boxed{\|\delta A\| / \|A\| \le \Delta}$$

$$\hat{A}\hat{x} = b$$

$$(A + \delta A)(x + \delta x) = b$$

$$\cancel{Ax} + \delta A x + A\delta x + \cancel{\delta A \delta x} = \cancel{b}$$

cancel ignore cancel

$$\delta A x = -A\delta x$$

$$\delta x = -A^{-1}\delta A x$$

$$\|\delta x\|_2 = \|A^{-1}\delta A x\|_2$$

$$\|\delta x\| \le \|A^{-1}\| \|\delta A\| \|x\|$$

$$\frac{\|\delta x\|}{\|x\|} \le \|A^{-1}\| \|\delta A\|$$

$$= \underbrace{\|A^{-1}\| \cdot \|A\|}_{\kappa(A)} \cdot \underbrace{\frac{\|\delta A\|}{\|A\|}}_{\le \Delta}$$

Solving Simple Linear Systems

- Solve $Dx = b$ if D is diagonal

$$x_i = b_i / d_{ii} \qquad O(n)$$

- Solve $Qx = b$ if Q is orthogonal

$$x = Q^T b$$

- Solve $Lx = b$ if L is lower-triangular

$$(U \Sigma V^T) x = b \qquad O(n^2)$$

$$U y = b \qquad O(n^2)$$

$$\Sigma z = y \qquad O(n)$$

$$\triangle \quad | = | \qquad L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$l_{11} x_1 = b_1$$

$$l_{21} x_1 + l_{22} x_2 = b_2 \quad \Big| \quad l_{22} x_2 = b_2 - l_{21} x_1$$