CS 450: Numerical Anlaysis

Lecture 14

Chapter 5 – Nonlinear Equations

Solving Systems of Nonlinear Equations

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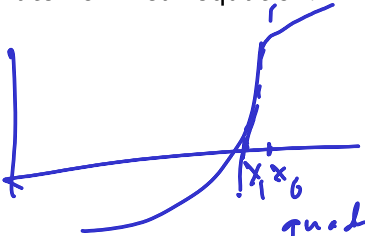
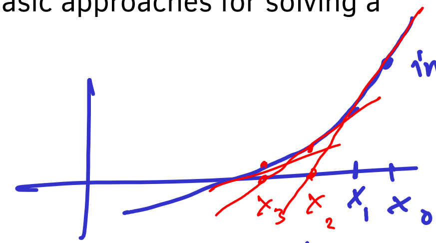
Review Solving a Nonlinear Equation

- Newton's and secant method provide basic approaches for solving a univariate nonlinear equation:

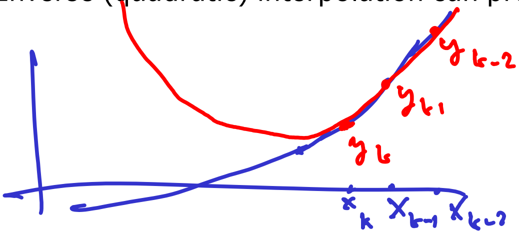


quadratic



superlinear

- Inverse (quadratic) interpolation can provide better convergence:



$$\begin{pmatrix} x_{k-2}, & y_{k-2} \\ x_{k-1}, & y_{k-1} \\ x_k, & y_k \end{pmatrix}$$

$y = p(x)$ $\quad \boxed{x = p(y)}$ $\quad x^* = p(0)$

Systems of Nonlinear Equations

- Given $\boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_m(\boldsymbol{x}) \end{bmatrix}$ for $\boldsymbol{x} \in \mathbb{R}^n$, seek $\boldsymbol{x}^* \in \mathbb{R}^n$ so that $\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$

$$f(x^*) = \begin{bmatrix} f_1(x^*) \\ \vdots \\ f_m(x^*) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- At a particular point $\boldsymbol{x}$, the *Jacobian* of $\boldsymbol{f}$, describes how $\boldsymbol{f}$ changes in a given direction of change in $\boldsymbol{x}$,

$$\boldsymbol{J_f}(\boldsymbol{x}) = \begin{bmatrix} \frac{df_1}{dx_1}(\boldsymbol{x}) & \cdots & \frac{df_1}{dx_n}(\boldsymbol{x}) \\ \vdots & & \vdots \\ \frac{df_m}{dx_1}(\boldsymbol{x}) & \cdots & \frac{df_m}{dx_n}(\boldsymbol{x}) \end{bmatrix}$$

Newton's method

$\hat{f}(x + \delta x) = 0$

$f(x) = -J_f(x)\,\delta x$

$\delta x = -J_f^{-1}(x)\, f(x)$

$$f(x + \delta x) \approx f(x) + J_f(x)\,\delta x = \hat{f}(x)$$

Multivariate Fixed-Point and Newton Iteration

- Fixed-point iteration $\boldsymbol{x}_{k+1} = \boldsymbol{g}(\boldsymbol{x}_k)$ achieves local convergence so long as $|\lambda_{\mathsf{max}}(\boldsymbol{J}_{\boldsymbol{g}}(\boldsymbol{x}^*))| < 1$:

spectral radius(A) = $|\lambda_{max}(A)|$

$\rho(A)$

- Newton's method corresponds to the fixed-point iteration

$$\boxed{\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x} - \underbrace{\boldsymbol{J}_{\boldsymbol{f}}^{-1}(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})}_{s}}$$

$g(x^*) = x^*$

$g(x^*) = x^*$ whenever $f(x^*) = 0$

$J_f(x)\, s = f(x)$ | $J_f(x^*)\, \underline{s} = f(x^*) = 0$

Convergence of Newton Iteration

- Newton's method $\overset{g(x)}{\text{achieves}}$ quadratic local convergence if $||J_f^{-1}(x^*)||$ is bounded:

$$e_k = x_k - x^* = g(x_{k-1}) - x^*$$

$$= x_{k-1} - J_f^{-1}(x_{k-1})f(x_{k-1}) - x^*$$

$$= -J_f^{-1}(x_{k-1})\Big[\overbrace{f(x_{k-1}) + J_f(x_{k-1})[x_{k-1} - x^*]}\Big]$$

$$||e_k|| \le ||J_f^{-1}(x_{k-1})||\,||f(x^*) - (\underbrace{f(x_{k-1}) + J_f(x_{k-1})(x^* - x_{k-1})})||$$

$$\le ||J_f^{-1}(x_{k-1})||\quad \underbrace{||x_{k-1} - x_*||_2^2} = O(||e_{k-1}||^2)$$

Convergence of Newton Iteration (II)

- Quadratic convergence is achieved when the Jacobian of a fixed-point iteration is zero at the solution:

$$J_g(x^*) = 0$$

$$g(x) = x - J_f(x)^{-1} f(x)$$

$$J_g(x) = I - \underbrace{\underbrace{J_f(x)^{-1} J_f(x)}_{I}}_{0} - \underbrace{f(x)}_{\substack{\downarrow \\ f(x^*) = 0}} \cdot \frac{d J_f(x)}{dx}$$

$$J_g(x^*) = 0$$

Estimating the Jacobian using Finite Differences

- To obtain $\boldsymbol{J_f}(\boldsymbol{x}_k)$ at iteration k, can use finite differences:

$n = 1 \quad x_k \in \mathbb{R}$

$f(x) \in \mathbb{R}^m \qquad J_f(x_k) \approx \frac{f(x+h) - f(x)}{h}$

$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \qquad j_i$ as ith column of $J_f(x_k)$

- How many function evaluations are generally needed?

$$j_i \approx \frac{f(x + he_i) - f(x)}{h}$$

evaluate

$$\left.\begin{array}{l} f(x) \\ f(x+he_1) \\ \vdots \\ f(x+he_n) \end{array}\right\} n+1$$