CS 450: Numerical Anlaysis

Lecture 5

Chapter 2 – Linear Systems

Solving Linear Systems

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Solving Triangular Systems

- $Lx = b$ **if** L **is lower-triangular is solved by forward substitution:**

$$
\begin{aligned}
l_{11}x_1 &= b_1 & & x_1 = b_1/l_{11} \\
l_{21}x_1 + l_{22}x_2 &= b_2 & \Rightarrow\quad & x_2 = (b_2 - l_{21}x_1)/l_{22} \\
l_{31}x_1 + l_{32}x_2 + l_{33}x_3 &= b_2 & & x_3 = (b_3 - l_{31}x_1 - l_{32}x_2)/l_{33} \\
&\vdots & & \vdots
\end{aligned}
$$

- **Computational complexity of forward/backward substitution:**

$T(n) = O(n^2)$

$\frac{n^2}{2}$ mults $\quad \frac{n^2}{2}$ additions

Solving Triangular Systems

- **Existence of solution to $Lx = b$:**

May not exist if some $L_{ii} = 0$, since then L is singular

- **Invertibility of L and existence of solution:**

sol'n may exist even if $L_{ii} = 0$, but is not unique

L is full rank if each $L_{ii} \neq 0$
(determinant shows this formally, since
$\det(L) = \prod_i L_{ii}$,
but can also see that kth column is not in the span of the last $n-k+1$, since $L_{kk} \neq 0$)

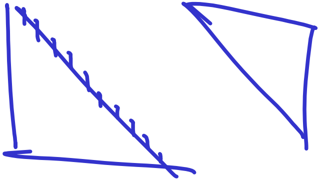
Properties of Triangular Matrices

- $XY = Z$ is lower triangular is X and Y are both lower triangular:

- L^{-1} is lower triangular if it exists:

LU Factorization

- An ***LU factorization*** consists of a unit-lower-triangular ***factor*** L and upper-triangular factor U such that $A = LU$:



$\frac{n(n-1)}{2}$

n^2 variables in $L, U \leftarrow \frac{n(n+1)}{2}$

n^2 variables in A

Gaussian Elimination

- **The LU factorization may not exist:**

  Consider matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 0 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & u_{22} \\ 0 & 0 \end{bmatrix}$$

Contradiction

$$\frac{3}{u_{22}} \qquad u_{22} = 0$$

$$l_{21} u_{12} = 3$$

$$2 l_{21} = 0$$

$$0$$

- **Permutation of variables enables us to transform the linear system so the LU factorization does exist:**

Gaussian Elimination Algorithm

- **Algorithm for factorization is derived from equations given by $A = LU$:**

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ & U_{22} \end{bmatrix}$$

$$A_{11} = L_{11} U_{11} \; ?$$

$$L_{11} U_{12} = A_{12}$$

$$L_{21} U_{11} = A_{21}$$

$$L_{21} U_{12} + L_{22} U_{22} = A_{22}$$

- **The kth column of L is given by the kth *elementary matrix* M_k:**

$$\underbrace{A_{22} - L_{21} U_{12}}_{\text{Schur complement}}$$

$$L_{21} U_{12} + L_{22} U_{22} = A_{22}$$

$$A_{22} - L_{21} U_{12} = L_{22} U_{22}$$

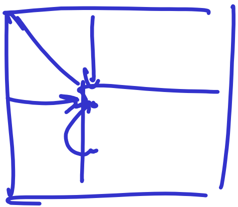
Elimination Matrices

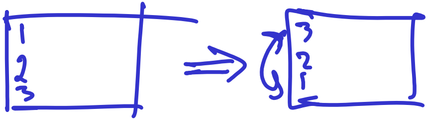
- **An elimination matrix M_k satisfies the following properties:**

Gaussian Elimination with Partial Pivoting

- ***Partial pivoting*** **permutes rows to make divisor u_{jj} is maximal at each step:**

$A = PLU \qquad P^T A = LU$





- **A row permutation corresponds to an application of a *row permutation matrix*** $P_{jk} = \underbrace{I - (e_j - e_k)(e_j - e_k)^T}$**:**

$P^T P = I$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P_{jk} = \begin{bmatrix} 1 & & \\ & \cdot & 1 \\ & 1 & \cdot \end{bmatrix}$$

Complete Pivoting and Error Bounds

- ***Complete pivoting*** **permutes rows and columns to make divisor u_{ii} is maximal at each step:**

Partial pivoting $|L|_{ij} \le 1$

- **For LU, the backward error δA, so that $\hat{L}\hat{U} = A + \delta A$, satisfies bound $|\delta A_{ij}| \le \epsilon(|\cancel{\hat{L}}| \cdot |\hat{U}|)_{ij}$:**

absolute value