CS 450: Numerical Anlaysis

Lecture 20
Chapter 7 Interpolation
Chebyshev Interpolation

Edgar Solomonik

Department of Computer Science
University of Illinois at Urbana-Champaign

April 4, 2018

Orthogonal Polynomials

- Recall that good conditioning for interpolation is achieved by constructing a well-conditioned Vandermonde matrix, which is the case when the columns (corresponding to each basis function) are orthonormal. To construct robust basis sets, we introduce a notion of *orthonormal functions*:

$$\langle f, g \rangle_w = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$\|f\| = \sqrt{\langle f, f \rangle_w} \longleftarrow \text{normalization (size measure)}$$

Legendre normalization $\Rightarrow$ by $f(1)$

end of interval

Legendre Polynomials

- The Gram-Schmidt orthogonalization procedure can be used to obtain an orthonormal basis with the same span as any given arbitrary basis:

$$[1, x, x^2, x^3]$$

$$f_1(x), f_2(x), \ldots$$

$$e_i(x) = f_i(x) - \sum_{j=1}^{i-1} \langle f_i(x), e_j(x) \rangle \cdot e_j(x)$$

- The Legendre polynomials are obtained by Gram-Schmidt on the monomial basis, with normalization done so $\hat{\phi}_i(1) = 1$ and $w(t) = \begin{cases} 1 : -1 \leq t \leq 1 \\ 0 : \text{otherwise} \end{cases}$

Basis Orthogonality and Conditioning

- To obtain perfectly conditioned Vandermonde system, want inner products of different columns to be zero:

A is orthogonal

$$A^T A = I$$

columns of A, $a_1 \dots a_n$

$$\langle a_i, a_j \rangle = \sum_{k=1}^{n} \varphi_i(t_k) \varphi_j(t_k)$$

$$= \delta_{ij}$$

- These inner products should be close to zero, if they are a suitable quadrature rule for our weighted functional inner product:

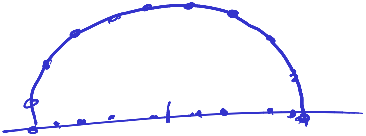
$$\int_{-\infty}^{\infty} w(t) \varphi_i(t) \varphi_j(t)\, dt \approx \sum_{k=1}^{n} \varphi_i(t_k) \varphi_j(t_k)$$

Chebyshev Basis

- *Chebyshev polynomials* $\phi_j(t) = \cos(j \arccos(t))$ and *Chebyshev nodes* $t_i = \cos\left(\frac{2i-1}{2n}\pi\right)$ provide a way to pick *nodes* $t_1, \ldots, t_n$ along with a basis, to yield perfect conditioning:

orthogonal polynomial w.r.t. $w(t) = 1/(1-t^2)$

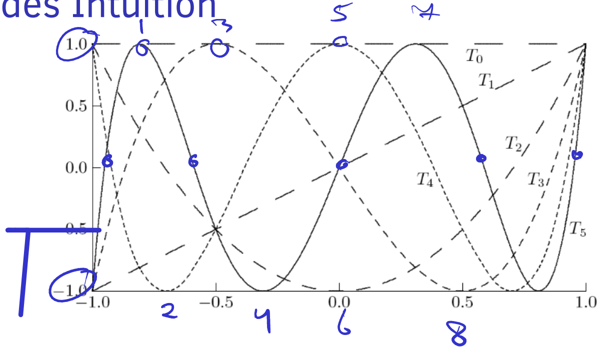
$$\phi_j(t_i) = \cos\left(j \frac{2i-1}{2n}\pi\right) = V_{ji}$$



$$\sum_{k=1}^{n} \phi_i(t_k)\phi_j(t_k) = \sum_{k=1}^{n} \cos\left(i \frac{2k-1}{2n}\pi\right)\cos\left(j \frac{2k-1}{2n}\pi\right) = 0 \text{ if } i \neq j$$

$$\sum_{k=1} V_{ik} V_{kj} = 0 \text{ if } i \neq j \qquad V^T V = D$$

Chebyshev Nodes Intuition



- Note *equi-alteration* property, successive extrema of T_k have opposite sign:

and equal magnitude

- Set of k Chebyshev nodes of are given by zeros of T_k

n-nodes get zeros T_n

use basis functions $T_0 \dots T_{n-1} \Rightarrow 1, x, \dots$

Orthogonal Polynomials and Recurrences

- The Newton polynomials could be obtained by a two-term recurrence
- Legendre and Chebyshev polynomials also satisfy three-term recurrence, for Chebyshev

$$\phi_{i+1}(t) = 2t\phi_i(t) - \phi_{i-1}(t)$$

$\varphi_0(t) = 1$

$\varphi_1(t) = x$

- In fact all orthogonal polynomials satisfy some recurrence of the form,

$$\varphi_{i+1}(t) = (\alpha_i + \beta_i t)\varphi_i(t) - \gamma_i \varphi_{i-1}(t)$$

Error in Interpolation

true function

Given degree n polynomial interpolant $\tilde{f}$ of f induction on n shows that $E(x) = f(x) - \tilde{f}(x)$ has n zeros $x_1, \ldots, x_n$ and there exist $y_1, \ldots, y_n$ such that

$[x_1, x_2] \ldots [x_{n-1}, x_n]$

interpolant

$$E(x) = \int_{x_1}^{x} \int_{y_1}^{w_0} \cdots \int_{y_n}^{w_{n-1}} f^{(n+1)}(w_n)\, dw_n \cdots dw_0 \qquad (1)$$

degree

inductive hypothesis

$$E(x) = \underbrace{E(x_1)}_{0} + \int_{x_1}^{x} \underbrace{E'(x)\, dx}$$

$$E(x_i) = 0$$

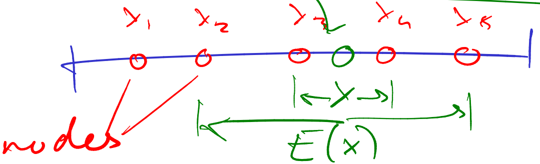
$$\exists\, y_i \in [x_{i-1}, x_i] \quad E'(y_i) = 0$$

$$E(x_i) - E(x_{i-1}) = \int_{x_{i-1}}^{x_i} E'(x)\, dx = 0$$

Interpolation Error Bounds

- The error bound implies that

$$|E(x)| \le \frac{\max_{s\in[x_1,x_n]} |f^{(n+1)}(s)|}{n!} \left|\prod_{i=1}^{n}(x - x_i)\right| \quad \text{for} \quad x \in [x_1, x_n]$$



equispaced

error minimum at $\frac{x_n + x_1}{2}$

- Letting $h = x_n - x_1$ (often also achieve same for h as the node-spacing $x_{i+1} - x_i$), we obtain

$$|E(x)| \le \frac{\max_{s\in[x_1,x_n]} |f^{(n+1)}(s)|}{n!} h^n = O(h^n) \quad \text{for} \quad x \in [x_1, x_n]$$

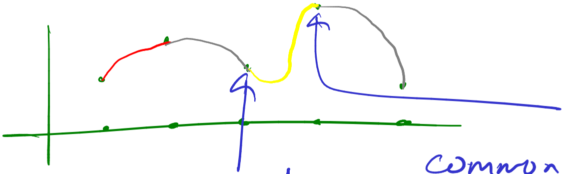
more nodes → increase n

decreasing h → nth order convergence

motivates piecewise polynomial interp.

Piecewise Polynomial Interpolation

- The kth piece of the interpolant is a polynomial in $[x_i, x_{i+1}]$



knots

enforce differentiability (once)

common to use cubic interpolants for each piece

- *Hermite* interpolation ensures consecutive interpolant pieces have same derivative at each *knot* x_i:

$f_i(x)$ is ith piece defined (x_i, x_{i+1})

$$\frac{\partial f_i}{\partial x}(x_{i+1}) = \frac{\partial f_{i+1}}{\partial x}(x_{i+1})$$

Spline Interpolation

- A *spline* is a $(k-1)$-time differentiable piecewise polynomial of degree k:

whole interpolant

cubic spline has $4(n-1)$ parameters

$k+1$ pieces/nodes

natural spline sets 2nd derivative to 0 at x_1 and x_n

$2(n-1)$ equations for interpolants, $n-1$ for each deriv.

- The resulting interpolant coefficients are again determined by an appropriate generalized Vandermonde system:

$$f_1(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

$$f_2(x) = \beta_1 + \beta_2 x + \dots$$

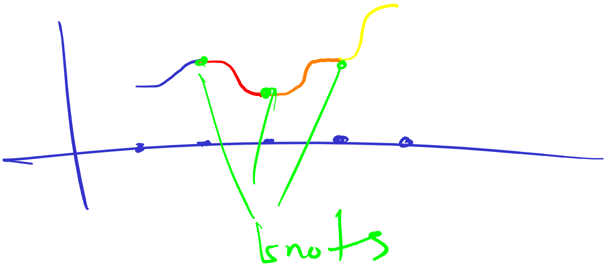
$$Vc = y$$

$$V \qquad c = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \\ \beta_1 \\ \vdots \\ \beta_4 \end{bmatrix}$$

$$2(n-1) + n-2 + n-2 = 4n-6$$

Lecture 21: Recap Interpolation

Piecewise interpolation



at each knot - both pieces interpolate

- smoothness, differentiability

Vandermonde systems

spline interpolant: degree k, $(k-1)$-times differentiable

e.g. $k=3$, twice-differentiable

natural spline - 2nd derivative at x_1, x_n is zero

B-Splines

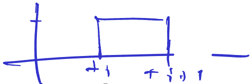
B-splines provide an effective way of constructing splines from a basis:

- The basis functions can be defined recursively with respect to degree.

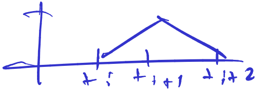
$$B_i^k = v_i^k B_i^{k-1} + (1-v_i^k) B_{i+1}^{k-1}$$

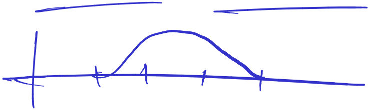
$$v_i^k(t) = \frac{t - t_i}{t - t_{i+k}}$$

$$B_i^0(t) = \begin{cases} 1 & \text{if } t \in [t_i, t_{i+1}] \\ 0 & \text{otherwise} \end{cases}$$



- The ith degree k polynomial piece is positive on $[t_i, t_{i+k+1}]$ and zero everywhere else





[compact support]

- All possible splines of degree k with notes $\{t_i\}_{i=1}^n$ can be represented in the basis.

so coefficients of neighboring splines are dependent $\rightarrow$ banded Vandermonde system