CS 450: Numerical Anlaysis

Lecture 25

Chapter 10 Boundary Value Problems for Ordinary Differential Equations

Fundamentals of ODE BVPs

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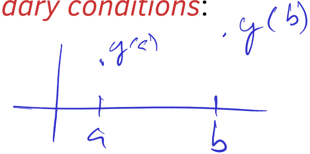
April 18, 2018

Boundary Value Problems for ODEs

- Often we seek to solve a differential equation that satisfies conditions on its values and derivatives on parts of the domain boundary. Consider a first order ODE $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$ with general *linear boundary conditions*:

$$\boxed{B_a} y(a) + \boxed{B_b} y(b) = \boxed{c}$$



if only y(x) specified as initial condition — Dirichlet

separable BCs, $B_b \neq 0$, $B_a \neq 0$

- High-order boundary conditions can be reduced to first-order like ODEs themselves:

Neumann BC, $y'(b) = f \Rightarrow u_2(b) = f$

$u_1 = y$

$u_2 = y'$

Boundary Value Problems for ODEs

nonhomogeneous ODE

- Can derive the solutions to a linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ from solutions to homogenous linear ODE $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t)$ IVPs:

$y_i(t)$ be sol'n to ODE $y'(t) = A(t)y(t)$ with $y_i(a) = e_i$

Let $Y(t) = [y_1(t) \; \dots \; y_n(t)]$

$Y(t) = I + \int_a^t A(s)\,Y(s)\,ds, \quad Y(a) = I$

$Y(a)\,y(a) = y(a)$, look for y of the form

$y(t) = Y(t)\,u(t)$ with $u(a) = y(a)$

$u(t) = y(a) + \int_a^t u'(s)\,ds$

Boundary Value Problems for ODEs

$y(t) = Y(t)u(t)$

- Can derive the solutions to a linear ODE BVP $\boldsymbol{y}'(t) = \boldsymbol{A}(t)\boldsymbol{y}(t) + \boldsymbol{b}(t)$ from solutions to homogenous linear ODE $\boldsymbol{y}' = \boldsymbol{A}(t)\boldsymbol{y}(t)$ IVPs:

$$B_a y(a) + B_b \boxed{y(b)} = c \qquad u(t) = y(a) + \int_a^t u'(s)\,ds$$

$$B_a Y(a)\, y(a) + B_b \underbrace{Y(b)\, y(a) + Y(b)\int_a^b u'(s)\,ds}_{y(b)} = c$$

$$\underbrace{\left[B_a Y(a) + B_b Y(b)\right]}_{Q} y(a) = \underbrace{c - B_b Y(b)\int_a^b u'(s)\,ds}_{b}$$

$$y'(t) = A(t)y(t) + b(t) = Y(t)\underbrace{Q^{-1}\left(c - B_b Y(b)\int_a^b u'(s)\,ds\right)}_{y_0} + Y(t)\int_a^t u'(s)\,ds$$

$$y'(t) = \cancel{A(t)y(t)} + b(t) = [Y(t)\, u(t)]'$$

$$= Y'(t)\, u(t) + Y(t)\, u'(t)$$

$$= \cancel{A(t) \underbrace{Y(t)\, u(t)}_{y(t)}} + Y(t)\, u'(t)$$

$$b(t) = Y(t)\, u'(t)$$

$$u'(t) = Y^{-1}(t)\, b(t)$$

$$y(t) = Y(t)u(t) = Y(t)\, y(a) + \int_a^t u'(s)\, ds$$

Linear ODE BVP Green's Function

$s(t) = Y(t) Q^{-1} c$

- We now express our solution (with form $\boldsymbol{y}(t) = \boldsymbol{Y}(t)(\boldsymbol{u}(a) + \int_a^t \boldsymbol{u}'(s)ds)$) in the form $\boldsymbol{y}(t) = \boldsymbol{s}(t) + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)\,ds$ where $\boldsymbol{G}$ is the *Green's function*:

$$y(t) = Y(t)Q^{-1}\left(c - B_b Y(b)\int_a^b Y^{-1}(s)\,b(s)\,ds\right) + Y(t)\int_a^t Y^{-1}(s)\,b(s)\,ds$$

$$Y(t)Q^{-1}\left[B_a Y(a)\int_a^t Y^{-1}(s)\,b(s)\,ds + B_b Y(b)\int_a^t Y^{-1}(s)\,b(s)\,ds\right]$$

$$Y(t)\,Q^{-1}\underbrace{\left[B_a Y(a) + B_b Y(b)\right]}_{Q}\int_a^t Y^{-1}(s)\,b(s)\,ds$$

$$y(t) = Y(t)\,Q^{-1}\left(c - B_a Y(a)\int_a^t Y^{-1}(s)\,b(s)\,ds + B_b Y(b)\int_t^b Y^{-1}(s)\,b(s)\,ds\right.$$

$$G(t,s) = Y(t)\,Q^{-1}\,I(s)\,Y^{-1}(s)\,b(s) \quad \text{where } I(s) = \begin{cases} B_a Y(a) & \text{if } s < t \\ -B_b Y(b) & \text{if } s > t \end{cases}$$

$G(t,s)$ does not depend on $b(t)$ and c

depends only on $A(t)$

(homogeneous ODE)

and B_a, B_b

Conditioning of Linear ODE BVPs

- For any given $\boldsymbol{b}(t)$ and $\boldsymbol{c}$, the solution to the BVP can be written in the form:

$$\boldsymbol{y}(t) = \boxed{\boldsymbol{\Phi}(t)}\boldsymbol{c} + \int_a^b \boldsymbol{G}(t,s)\boldsymbol{b}(s)ds$$

$Y(t)Q^{-1}$

- The absolute condition number of the BVP is $\kappa = \max\{||\boldsymbol{\Phi}||_\infty, ||\boldsymbol{G}||_\infty\}$:

how much does sol'n change if we perturb b or c, $||\hat{y} - y||_\infty \leq \kappa ||\text{change}||$

Shooting Method for ODE BVPs

- For linear ODEs, we constructed solutions from IVP solutions in $\boldsymbol{Y}(t)$, which kth suggests a method for solving BVPs by reduction to IVPs:

Shooting method works by iteratively guessing initial condition $y'(a)$, then solving IVP, so satisfy ODE, but not BC, $\|B_b \hat{y}^{(k)}(b) - (c - B_a y(a))\|$

- *Multiple shooting* employs the shooting method over subdomains:

ODE IVP may be unstable even when the BVP is well-conditioned.

find $y(a)$: $g(x) = 0$

IVP soln with $y(a) = x$

$$g(x) = B_b y_x(b) + B_a x - c$$

Multiple shooting subdivides $[a,b]$ into subintervals and performs shooting on each one

Finite Difference Methods

- Rather than solve a sequence of IVPs that satisfy the ODEs until they (approximately) satisfy boundary conditions, we can refine an approximation that satisfies the boundary conditions, until it satisfies the ODE:

Approximate differential operator with finite differences

$y'(t) = f(t, y(t))$, discretize on $t_1, \dots, t_n$

$y'(t_k) \approx \dfrac{y(t_{k+1}) + 2y(t_k) - y(t_{k-1})}{2h} \rightarrow \dfrac{\hat{y}_{k+1} + 2\hat{y}_k - \hat{y}_{k-1}}{2h}$ $\quad y(t_i) \approx \hat{y}_i$

$t_i = t_0 + ih$

- Convergence to solution is obtained with decreasing step size h so long as the method is consistent and stable:

consistency means that $h \to 0$, $\hat{y}_k \to y(t_k)$

stability, insensitivity to perturbation